Computation of the Riemann Function for the Operator $\partial^n/\partial x_1 \partial x_2 \cdots \partial x_n + a(x_1, x_2, \cdots, x_n)$

By H. M. Sternberg and J. B. Diaz

1. Introduction. Riemann's method [1] is, in principle, well suited for the numerical solution of boundary value problems for the hyperbolic differential equation

$$\partial^2 u/\partial x_1 \partial x_2 + a(x_1, x_2)u = F(x_1, x_2). \tag{1.1}$$

One first finds the Riemann function, which is the solution of a homogeneous adjoint equation subject to simple boundary conditions that are independent of the given boundary data. The solution of (1.1), for any appropriate boundary data, can then be obtained by evaluating a definite integral, where the Riemann function and the boundary data appear in the integrand.

The main obstacle to the use of this method for numerical computation has been the difficulty in finding the Riemann function. It was pointed out in a 1947 paper by Cohn [2] that, aside from $a(x_1, x_2) = \text{constant}$ and $a(x_1, x_2) = k(k-1) \cdot (x_1 + x_2)^{-2}$, which was treated by Riemann, there are very few cases where expressions for the Riemann function have been obtained. One can, of course, use the Picard method of successive approximations, but this is usually not practical. In this paper we derive a simple recurrence formula for the Riemann function, for the case where $a(x_1, x_2)$ can be expanded in a Taylor series about the point where the solution is sought.

We consider here the Riemann function for the operator L in the n dimensional analogue of (1.1),

$$L(u) = \partial^n u/\partial x_1 \partial x_2 \cdots \partial x_n + a(x_1, x_2, \cdots, x_n)u = F(x_1, x_2, \cdots, x_n). \tag{1.2}$$

It was shown by Bianchi [3], [4], [5] and Niccoletti [6], who treated the more general linear hyperbolic equation containing pure mixed derivatives of all orders through the nth, that Riemann's method can be extended to this case. The solutions of the characteristic value and the Cauchy problem for (1.2) are discussed in detail in [7]. The Riemann function v for the operator L in (1.2) can be defined by the integral equation

$$\begin{aligned} &v(x_1, x_2, \cdots, x_n : \beta_1, \beta_2, \cdots, \beta_n) \\ &\quad = 1 - \int_{x_1}^{\beta_1}\int_{x_2}^{\beta_2} \cdots \int_{x_n}^{\beta_n} a(\xi_1, \xi_2, \cdots, \xi_n) \\ &\qquad \cdot v(\xi_1, \xi_2, \cdots, \xi_n : \beta_1, \beta_2, \cdots, \beta_n)\, d\xi_1 d\xi_2 \cdots d\xi_n. \end{aligned} \tag{1.3}$$

The solution of the characteristic value problem for (1.2) can, for example, be written

$$\begin{aligned} u(\beta_1, \beta_2, \cdots, \beta_n) &= \bar{u}(\beta_1, \beta_2, \cdots, \beta_n) \\ &+ \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} \cdots \int_{\alpha_n}^{\beta_n} v(x_1, x_2, \cdots, x_n : \beta_1, \beta_2, \cdots, \beta_n) \\ &\cdot [F - a\bar{u}](x_1, x_2, \cdots, x_n)\, dx_1 dx_2 \cdots dx_n, \end{aligned} \tag{1.4}$$

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where $\bar{u}$ is the solution of $\partial^n u/\partial x_1 \partial x_2 \cdots \partial x_n = 0$ satisfying the data given on the hyperplanes $x_i = \alpha_i$.

The present discussion is confined to the case where the function $a(x_1, x_2, \cdots, x_n)$ is expandable about the point $(\beta_1, \beta_2, \cdots, \beta_n)$ in a Taylor series valid in the region $|x_i - \beta_i| < r_i'$. This is equivalent to the assumption that $a(x_1, x_2, \cdots, x_n)$ is an analytic function of n complex variables. The Taylor series for v then has the same region of convergence as the series for a (Section 4). Less stringent regularity conditions can be applied to the function F and the given boundary data.

2. The Recurrence Formula. To fix the ideas, the recurrence formula for v will be obtained for the one dimensional case first. Suppose v is expandable in the Taylor series

$$v = 1 + \sum_{i=1}^{\infty} f_i(x - \beta)^i, \qquad f_i = \frac{1}{i!} \frac{\partial^i v}{\partial x^i}\bigg]_{x=\beta}. \tag{2.1}$$

From (1.3),

$$\begin{aligned} \partial v/\partial x]_{x=\beta} &= [a(x)v(x:\beta)]_{x=\beta} = a(\beta), \\ \partial^2 v/\partial x^2]_{x=\beta} &= \partial(av)/\partial x]_{x=\beta} = [a\partial v/\partial x + v\partial a/\partial x]_{x=\beta}, \\ \partial^i v/\partial x^i]_{x=\beta} &= \partial^{i-1}(av)/\partial x^{i-1}]_{x=\beta} = a(\beta)\partial^{i-1} v/\partial x^{i-1}]_{x=\beta} + \cdots. \end{aligned} \tag{2.2}$$

This method for finding the coefficient f_i in (2.1), from the previous ones, was proposed by du Bois Reymond [8] for the two dimensional case.

Since all the derivatives $\partial^i a/\partial x^i$ are required in (2.2), one might as well use the essentially identical, but simpler, procedure which consists of taking

$$a(x) = \sum_{i=0}^{\infty} a_i(x - \beta)^i, \qquad a_i = \frac{1}{i!} \frac{\partial^i a}{\partial x^i}\bigg]_{x=\beta}, \tag{2.3}$$

together with (2.1), inserting these in (1.3) and equating coefficients. The result is the recurrence formula

$$if_i = \sum_{k=0}^{i-1} a_{i-1-k} f_k. \tag{2.4}$$

For the higher dimensional cases, the analogue of (2.4) is a convenient and practical way to get the Riemann function. Its merit lies in the fact that the same coefficients are not computed over and over again, as in the method of successive approximations.

For the Taylor series expansion of v in the n dimensional case let

$$b(x_1, x_2, \cdots, x_n) = (-1)^{n+1} a(x_1, x_2, \cdots, x_n), \tag{2.5}$$

so that, from (1.3)

$$v_{,12\cdots n} = bv, \quad v(x_1, x_2, \cdots, x_{i-1}, \beta_i, x_{i+1}, \cdots, x_n : \beta_1, \beta_2, \cdots, \beta_n) = 1. \tag{2.6}$$

Let

$$b = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} b_{i_1 i_2 \cdots i_n}(x_1 - \beta_1)^{i_1}(x_2 - \beta_2)^{i_2} \cdots (x_n - \beta_n)^{i_n}, \tag{2.7}$$

and

$$(2.8)\qquad v = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_n=0}^{\infty} f_{i_1 i_2\cdots i_n}(x_1-\beta_1)^{i_1}(x_2-\beta_2)^{i_2}\cdots(x_n-\beta_n)^{i_n},$$

where

$$(2.9)\qquad b_{i_1 i_2\cdots i_n} = \frac{1}{i_1!\, i_2!\cdots i_n!}\left.\frac{\partial^{i_1+i_2+\cdots+i_n} b}{\partial x_1^{i_1}\partial x_2^{i_2}\cdots\partial x_n^{i_n}}\right]_{x_k=\beta_k,\, k=1,2,\cdots,n},$$

and

$$(2.10)\qquad f_{i_1 i_2\cdots i_n} = \frac{1}{i_1!\, i_2!\cdots i_n!}\left.\frac{\partial^{i_1+i_2+\cdots+i_n} v}{\partial x_1^{i_1}\partial x_2^{i_2}\cdots\partial x_n^{i_n}}\right]_{x_k=\beta_k,\, k=1,2,\cdots,n}$$

Here, from (1.3), $f_{00\cdots0} = 1$, and any other $f_{i_1 i_2\cdots i_n}$ vanishes if at least one of the i's is zero.

The multiplication of (2.7) and (2.8) and the insertion of the result in (2.6) leads to the recurrence formula for the coefficients in (2.8), namely,

$$(2.11)\qquad i_1 i_2\cdots i_n f_{i_1 i_2\cdots i_n} = \sum_{k_1=0}^{i_1-1}\sum_{k_2=0}^{i_2-1}\cdots\sum_{k_n=0}^{i_n-1} b_{i_1-1-k_1,\, i_2-1-k_2,\cdots,\, i_n-1-k_n} f_{k_1 k_2\cdots k_n}\,.$$

Once the Riemann function, corresponding to a particular function $a(x_1, x_2, \cdots, x_n)$, is known, the characteristic boundary value problem, for example, is reduced to one of evaluating a multiple integral [see (1.4)]. The numerical computation proceeds as follows: For a fixed point $(\beta_1, \beta_2, \cdots, \beta_n)$ at which the solution is sought, the coefficients $f_{i_1 i_2\cdots i_n}$ are calculated, to any desired accuracy, from the recurrence formula (2.11). In general, these coefficients will be functions of $\beta_1, \beta_2, \cdots, \beta_n$. The grid points $(x_1, x_2, \cdots, x_n)$ to be used in evaluating the integral in (1.4) are then selected. For each of these points $v(x_1, x_2, \cdots, x_n : \beta_1, \beta_2, \cdots, \beta_n)$ is calculated, using (2.8), and stored. The solution of (1.2) at the point $(\beta_1, \beta_2, \cdots, \beta_n)$ is then obtained for any appropriate set of boundary data, by the numerical integration of (1.4), using the stored values of v. The same v can also be used for the solution of the Cauchy problem for (1.2). Here, too, the solution can be expressed in terms of an integral containing v [7].

3. Examples.

A. To test the method, the recurrence formula (2.11) is used here to calculate the Riemann function v for a case where v is known explicitly, namely,

$$(3.1)\qquad L(u) = \partial^2 u/\partial x_1\partial x_2 - (2/(x_1+x_2)^2)u.$$

Here, the Riemann function, which must satisfy

$$(3.2)\qquad \partial^2 v/\partial x_1\partial x_2 - [2/(x_1+x_2)^2]v = 0,\quad v(x_1, \beta_2 : \beta_1, \beta_2) = v(\beta_1, x_2 : \beta_1, \beta_2) = 1,$$

is

$$(3.3)\qquad v(x_1, x_2 : \beta_1, \beta_2) = 2\frac{(x_2+\beta_1)(x_1+\beta_2)}{(x_1+x_2)(\beta_1+\beta_2)} - 1.$$

For comparison with (3.3), (2.11) and (2.7) were used to calculate $v(x_1, x_2 : 1, 1)$

TABLE I

The coefficients $f_{i_1 i_2}$ in the expansion of the Riemann function $v(x_1, x_2{:}1, 1)$ for the operator $\partial^2/\partial x_1 \partial x_2 - 2/(x_1 + x_2)^2$

i_2 \ i_1	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	0	.50000000	−.25000000	.12500000	−.06250000	.03125000	−.01562500	.00781250	−.00390625	.00195312	−.00097656
2	0	−.25000000	.25000000	−.18750000	.12500000	−.07812500	.04687500	−.02734375	.01562500	−.00878906	.00488281
3	0	.12500000	−.18750000	.18750000	−.15625000	.11718750	−.08203125	.05468750	−.03515625	.02197266	−.01342773
4	0	−.06250000	.12500000	−.15625000	.15625000	−.13671875	.10937500	−.08203125	.05859375	−.04028320	.02685547
5	0	.03125000	−.07812500	.11718750	−.13671875	.13671875	−.12304687	.10253906	−.08056640	.06042480	−.04364014
6	0	−.01562500	.04687500	−.08203125	.10937500	−.12304687	.12304787	−.11279297	.09667969	−.07855224	.06109619
7	0	.00781245	−.02734375	.05468750	−.08203125	.10253906	−.11279297	.11279297	−.10473633	.09164429	−.07637024
8	0	−.00390625	.01562500	−.03515625	.05859375	−.08056640	.09667969	−.10473633	.10473633	−.09819031	.08728027
9	0	.00195312	−.00878906	.02197266	−.04028320	.06042480	−.07855224	.09165529	−.09819031	.09819031	−.09273529
10	0	−.00097656	.00488281	−.01342773	.02685547	−.04364014	.06109619	−.07637024	.08728027	−.09273529	.09273529

TABLE II

The nth partial sums and the exact values of the Riemann function $v(x_1, x_2 : 1, 1)$ for the operator $\partial^2/\partial x_1 \partial x_2 - 2/(x_1 + x_2)^2$.

	(x_1, x_2)			
	(.8, .8)	(.6, .6)	(.4, .4)	(.2, .2)
$n = 1$	1.020000	1.080000	1.180000	1.320000
2	1.024400	1.118400	1.320400	1.678400
3	1.024932	1.129408	1.390708	1.952832
4	1.024992	1.132326	1.423464	2.151078
5	1.024999	1.133078	1.438273	2.290932
6	1.025000	1.133269	1.444859	2.388318
7	1.025000	1.133317	1.447759	2.455571
8	1.025000	1.133329	1.449027	2.501749
9	1.025000	1.133332	1.449579	2.533320
10	1.025000	1.133333	1.449818	2.554834
Exact value	1.025000	1.133333	1.450000	2.600000

for several points (x_1, x_2). In two dimensions, the recurrence formula (2.11) is

$$i_1 i_2 f_{i_1 i_2} = \sum_{k_1=0}^{i_1-1} \sum_{k_2=0}^{i_2-1} b_{i_1-1-k_1, i_2-1-k_2} f_{k_1 k_2}, \tag{3.4}$$

$$f_{00} = 1, \qquad f_{0j} = f_{j0} = 0 \quad \text{for} \quad j \neq 0.$$

For $b = 2/(x_1 + x_2)^2$ the coefficients in (2.7) are

$$b_{ij} = 2(-1)^{i+j}(i + j + 1)!/[i!j!(\beta_1 + \beta_2)^{2+i+j}]. \tag{3.5}$$

The coefficients $f_{i_1 i_2}$ in the expansion (2.8) for $v(x_1, x_2{:}1, 1)$ are listed for $i_1, i_2 \leqq 10$ in Table I. In Table II, the exact values of $v(x_1, x_2{:}1, 1)$, obtained with (3.3), are listed for several values of (x_1, x_2), together with the nth partial sums calculated from

$$v^{(n)} = 1 + \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} f_{i_1 i_2}(x_1 - 1)^{i_1}(x_2 - 1)^{i_2}. \tag{3.6}$$

B. It is evident from (1.4), and the analogous formula for the solution of the Cauchy problem [7], that the value of the solution of (1.2) at the single point $(\beta_1, \beta_2, \cdots, \beta_n)$ can be found, for any appropriate boundary data, if the Riemann function v is known for that point. Consider the operator

$$\partial^n/\partial x_1 \partial x_2 \cdots \partial x_n + a(\sigma), \tag{3.7}$$

where

$$\sigma = x_1 x_2 \cdots x_n . \tag{3.8}$$

To find the Riemann function v for $(\beta_1, \beta_2, \cdots, \beta_n) = (0, 0, \cdots, 0)$ let

$$b(\sigma) = (-1)^{n+1} a(\sigma), \tag{3.9}$$

$$b = \sum_{i=0}^{\infty} b_i \sigma^i, \tag{3.10}$$

and

$$v = 1 + \sum_{i=1}^{\infty} f_i \sigma^i. \tag{3.11}$$

For this special case, the recurrence formula (2.11) becomes

$$i^n f_i = \sum_{k=0}^{i-1} b_{i-1-k} f_k . \tag{3.12}$$

Now suppose

$$b = \frac{K_1}{1 - K_2 \sigma}, \tag{3.13}$$

where K_1 and K_2 are constants. Here, the coefficients in (3.10) are

$$b_i = K_1 K_2^i, \qquad | K_2 \sigma | < 1. \tag{3.14}$$

From (3.14) and the recurrence formula (3.12),

$$i^n f_i = [K_1 + K_2(i-1)^n] f_{i-1} . \tag{3.15}$$

Now, from (3.15),

$$\begin{aligned} (i-1)^n f_{i-1} &= [K_1 + K_2(i-2)^n] f_{i-2} , \\ (i-2)^n f_{i-2} &= [K_1 + K_2(i-3)^n] f_{i-3} , \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ 2f_2 &= (K_1 + K_2) f_1 , \\ f_1 &= K_1 . \end{aligned} \tag{3.16}$$

Multiplication of the terms in (3.15) and (3.16) leads to the coefficients in the expansion for $v(x_1, x_2, \cdots, x_n : 0, 0, \cdots, 0)$, namely,

$$f_i = K_1(K_1 + K_2)(K_1 + 2^n K_2) \cdots (K_1 + (i-1)^n K_2)/[i!]^n. \tag{3.17}$$

The substitution $v = v(\sigma)$ in (2.6), with $a = (-1)^{n+1} K_1/(1 - K_2\sigma)$, results in an ordinary differential equation for which

$$v = 1 + \sum_{i=1}^{\infty} f_i x^i, \tag{3.18}$$

with f_i given by (3.17), is a particular solution. For example, in two dimensions (3.18) is the solution of the hypergeometric equation

$$\sigma \frac{d^2 v}{d\sigma^2} + \frac{dv}{d\sigma} - \frac{K_1}{1 - K_2 \sigma} v = 0, \qquad v(0) = 1, \tag{3.19}$$

and for $n = 3$, (3.18) is the solution of

$$\sigma^2 \frac{d^3 v}{d\sigma^3} + 3\sigma \frac{d^2 v}{d\sigma^2} + \frac{dv}{d\sigma} + \frac{K_1}{1 - K_2 \sigma} v = 0, \qquad v(0) = 1. \tag{3.20}$$

4. The Convergence of the Expansion for the Riemann Function. The series (2.8) for the Riemann function will converge in the open region about

$(\beta_1, \beta_2, \cdots, \beta_n)$ where (2.7), the expansion for b, converges. This can be shown as follows: Suppose that (2.7) converges in the region $|x_i - \beta_i| \leqq r_i'$. Let M be the maximum $|b_{i_1 i_2 \cdots i_n}|$ in this region and let

$$B = \frac{M}{\left[1 - \frac{x_1 - \beta_1}{r_1}\right]\left[1 - \frac{x_2 - \beta_2}{r_2}\right] \cdots \left[1 - \frac{x_n - \beta_n}{r_n}\right]}, \tag{4.1}$$

where $0 < r_i < r_i'$. Denote the expansion for B in $|x_i - \beta_i| < r_i$ by

$$B = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} B_{i_1 i_2 \cdots i_n}(x_1 - \beta_1)^{i_1}(x_2 - \beta_2)^{i_2} \cdots (x_n - \beta_n)^{i_n}, \tag{4.2}$$

and confine the discussion to the region $|x_i - \beta_i| < r_i < r_i'$. Here (4.2) dominates (2.7), i.e.,

$$B_{i_1 i_2 \cdots i_n} \geqq |b_{i_1 i_2 \cdots i_n}|. \tag{4.3}$$

The solution of

$$\begin{gathered} \partial^n V / \partial x_1 \partial x_2 \cdots \partial x_n = BV, \\ V(x_1, x_2, \cdots, x_{i-1}, \beta_i, x_{i+1}, \cdots, x_n : \beta_1, \beta_2, \cdots, \beta_n) = 1 \end{gathered} \tag{4.4}$$

can, in its domain of convergence, be written

$$V = 1 + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \cdots \sum_{i_n=1}^{\infty} F_{i_1 i_2 \cdots i_n}(x_1 - \beta_1)^{i_1}(x_2 - \beta_2)^{i_2} \cdots (x_n - \beta_n)^{i_n}. \tag{4.5}$$

The coefficients in (4.5) are obtained from (2.11) as

$$i_1 i_2 \cdots i_n F_{i_1 i_2 \cdots i_n} = \sum_{k_1=0}^{i_1-1} \sum_{k_2=0}^{i_2-1} \cdots \sum_{k_n=0}^{i_n-1} B_{i_1-1-k_1, i_2-1-k_2, \cdots, i_n-1-k_n} F_{k_1 k_2 \cdots k_n}. \tag{4.6}$$

Since the coefficients $B_{i_1 i_2 \cdots i_n}$ are positive, it follows from (4.3) and (4.6) that (4.5) dominates (2.8), i.e.,

$$F_{i_1 i_2 \cdots i_n} \geqq |f_{i_1 i_2 \cdots i_n}|. \tag{4.7}$$

To find the region of convergence of (4.5), consider the constant coefficient equation

$$\partial^n V / \partial \xi_1 \partial \xi_2 \cdots \partial \xi_n = MV. \tag{4.8}$$

Bianchi [3] obtained the solution of (4.8) as

$$\begin{aligned} &V(\xi_1, \xi_2, \cdots, \xi_n : \beta_1, \beta_2, \cdots, \beta_n) \\ &\qquad = 1 + \sum_{j=1}^{\infty} [M^j[(\xi_1 - \beta_1)(\xi_2 - \beta_2) \cdots (\xi_n - \beta_n)]^j / (j!)^n]. \end{aligned} \tag{4.9}$$

This result, (4.9), is also evident as a special case of Example B, Section 3, with $K_2 = 0$. The transformations

$$\xi_i - \beta_i = -r_i \ln \left[1 - \frac{x_i - \beta_i}{r_i}\right] = \sum_{k=1}^{\infty} \frac{(x_i - \beta_i)^k}{k r_i^{k-1}} \tag{4.10}$$

convert (4.8) into (4.4). From (4.9) and (4.10), the solution of (4.4) can be written

$$(4.11) \qquad V = 1 + \sum_{j=1}^{\infty} \left\{ (1/j!)^n \left[(-1)^n M r_1 r_2 \cdots r_n \prod_{i=1}^{n} \ln \left(1 - \frac{x_i - \beta_i}{r_i} \right) \right]^j \right\}.$$

Since (4.5) is equivalent to (4.11), it converges in the region $|x_i - \beta_i| < r_i$. The series (2.8) for the Riemann function, since it is dominated by (4.5), must therefore converge in the region $|x_i - \beta_i| < r_i < r_i'$.

U. S. Naval Ordnance Laboratory
White Oak, Maryland and
Institute for Fluid Dynamics and
Applied Mathematics
University of Maryland
College Park, Maryland

1. B. RIEMANN, "Über die fortpflanzung ebener luftwellen von endlicher schwingungsweite," *Gesammelte Mathematische Werke*, Leipzig, 1892, pp. 156–181.

2. H. COHN, "The Riemann function for $\partial^2 u/\partial x \partial y + H(x + y)u = 0$," *Duke Math. J.*, v. 14, 1947, pp. 297–304. MR **9**, 38.

3. L. BIANCHI, "Il metodo di Riemann esteso alla integrazione della equazione: $\partial^n u/\partial x_1 \partial x_2 \cdots \partial x_n = Mu$," *Rend. Accad. Lincei Ser.* 5, v. 4, 1895, pp. 8–18.

4. L. BIANCHI, "Sulla estensione del metodo di Riemann alle equazioni lineari alle derivate parziali d'ordine superiore," *Rend. Accad. Lincei Ser.* 5, v. 4, 1895, pp. 89–99.

5. L. BIANCHI, "Sulla estensione del metodo di Riemann alle equazioni lineari a derivate parziali d'ordine superiore Nota III," *Rend. Accad. Lincei Ser.* 5, v. 4, 1895, pp. 133–142.

6. O. NICCOLETTI, "Sull'estensione del metodo di Riemann alle equazioni lineari a derivate parziali d'ordine superiore," *Rend. Accad. Lincei Ser.* 5, v. 4, 1895, pp. 330–337.

7. H. M. STERNBERG & J. B. DIAZ, "The solution of the characteristic and the Cauchy boundary value problems for the Bianchi partial differential equation in n independent variables by a generalization of Riemann's method," NavWeps Report 7264, Oct., 1960.

8. P. DU BOIS REYMOND, "Über lineare partielle differentialgleichungen zweiter ordnung," *J. für Math.*, v. 104, Heft 3, 1889, pp. 241–301.