A Numerical Survey of the Floors of Various Hilbert Fundamental Domains

By Harvey Cohn

1. Introduction. From the purely computational point of view we are considering a real positive function $S_2 = f(R_1, R_2, S_1)$ defined (in (3.15) below) for values R_1, R_2, and S_1 varying on a parallelopiped. The function f is composed of a large and *undetermined* number (possibly thousands!) of analytic pieces. The object is to find the minimum of f and to estimate the number of pieces which constitute f. What we do is probably the easiest thing: We subdivide the parallelopiped by a regular three-dimensional grid and scan for min S_2 as well as the number of pieces in the function f.

The function f arises here in an interesting context, however, since it represents part of the boundary (called the "floor") of the fundamental domain $\boldsymbol{R}$ for Hilbert's modular group for certain quadratic fields of unique factorization. (We assume some knowledge of factorization theory [1] but we summarize Siegel's theory of these fundamental domains [2], [5], for easy reference.) It is important to know the minimum of S_2 (and the optimal "low-point") because of applications to relative-quadratic fields [2, §6], [4] and it is important to know the number of pieces constituting f as a clue to the topological structure of $\boldsymbol{R}$ under boundary identifications.

The most important single result is the Main Theorem of §5 (below). It states that among quadratic fields of unique factorization, with the only exception of the field of $5^{1/2}$, no fundamental domain of the corresponding Hilbert modular group can have a "simple floor" or a floor consisting of "a single piece" (if we ignore translates; these terms will be explained more precisely below). The proof is based on counterexamples which were yielded by a computational search process.

The computer here serves as a crude instrument since it may fail to discern all pieces of the surface being examined or it may overestimate the "low-point" on the surface. Still we must not discount the possibility that a thorough round-off analysis could establish an error estimate for the scanning, which could be used, provided vast amounts of computing time became available.

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2. Summary of Theory. We consider the quadratic field $\boldsymbol{K}$ generated by $k^{1/2}$ where k is a square-free integer > 1 and where the field in question has class-number unity. We take 12 cases:

$$k = 2, 3, 5, 6, 7, 11, 13, 14, 17, \cdots, 21, 29, 33, \cdots$$

We let Greek letters denote integers in $\boldsymbol{K}$ and "primed" Greek letters denote con-

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jugates, while lower-case Roman letters denote rational integers. We let $\boldsymbol{O}$ denote the ring of integers in $\boldsymbol{K}$; thus $\boldsymbol{O}$ is the set of

$$\alpha = x + y\omega, \qquad \omega = [(c-1) + k^{1/2}]/c \tag{2.1}$$

and $c = 2$ if $k \equiv 1 \pmod 4$ while $c = 1$ otherwise. We call α *totally positive* ($\alpha \gg 0$) when $\alpha > 0$ and $\alpha' > 0$, and we let $\epsilon_+ (>1)$ denote the fundamental totally positive unit. (The identification of ϵ_+ with the fundamental unit ϵ_0 or with ϵ_0^2 is shown in Table 1 below.)

Consider now the bicomplex upper half planes $\boldsymbol{U} \times \boldsymbol{U}$

$$Z = X + iY, \qquad Z' = X + iY' \qquad (Y > 0,\ Y' > 0), \tag{2.2}$$

where continuous variables are denoted by capital Roman letters. The primes here denote formal relations so that norm $\boldsymbol{N}$ and trace $\boldsymbol{S}$ can apply equally well to the complex variables as to the algebraic numbers. Thus $\boldsymbol{N}(\alpha Y) = \alpha\alpha' YY'$, $\boldsymbol{S}(\alpha Y) = \alpha Y + \alpha' Y'$, etc. We call

$$\| Z \| = \boldsymbol{N}(X^2 + Y^2) = (X^2 + Y^2)(X'^2 + Y'^2) \tag{2.3}$$

and we call the *height*

$$\boldsymbol{h}(Z, Z') = \boldsymbol{N}(Y) = YY'. \tag{2.4}$$

The Hilbert modular group $\boldsymbol{H}$ is the group of transformations of $\boldsymbol{U} \times \boldsymbol{U}$ onto itself

$$Z \to (\alpha Z + \beta)/(\gamma Z + \delta), \qquad Z' \to (\alpha' Z' + \beta')/(\gamma' Z' + \delta') \tag{2.5}$$

where the determinant is a totally positive unit or

$$\alpha\delta - \beta\gamma = \epsilon \gg 0. \tag{2.6}$$

The indicated transformations divide the space $\boldsymbol{U} \times \boldsymbol{U}$ into equivalence classes and a fundamental domain $\boldsymbol{R}$ for $\boldsymbol{H}$ is a minimal subset of $\boldsymbol{U} \times \boldsymbol{U}$ representing each equivalence class (exactly once). The choice of $\boldsymbol{R}$ is not definite but, in principle, one desires to have a conveniently small number of bounding (three dimensional) manifolds. The exact number is of such manifolds still unknown; all we show is that this number becomes quite large as k increases, according to the computations.

An important subgroup of $\boldsymbol{H}$ is $\boldsymbol{H}_\infty$ which leaves fixed the point at ∞ ($Z = \infty$, $Z' = \infty$). It consists of the transformations

$$Z \to \epsilon Z + \beta, \qquad Z' \to \epsilon' Z + \beta' \tag{2.7}$$

where $\epsilon \gg 0$ is a unit. A fundamental domain for $\boldsymbol{H}_\infty$ is rather elementary (see [2, §4]); it is seen to be given by $\boldsymbol{R}_\infty$ which is the cartesian product of a wedge and parallelogram:

$$\epsilon_+^{-1} \leqq Y'/Y \leqq \epsilon_+, \tag{2.8}$$

$$-\tfrac{1}{2} \leqq R_1 \leqq \tfrac{1}{2}, \qquad -\tfrac{1}{2} \leqq R_2 \leqq \tfrac{1}{2}, \tag{2.9a}$$

where we use the computationally convenient variables R_1, R_2 defined by

$$X = R_1 + \omega R_2, \qquad X' = R_1 + \omega' R_2. \tag{2.9b}$$

Of course boundary points are identified through (2.7) in obvious fashion.

It also can be shown that of all points in each equivalence class created by (2.5) there exist (several) points where $\boldsymbol{h}(Z, Z')$ is maximized. Such points are characterized by the fact they belong in the domain given by (see [2, §4]),

$$\boldsymbol{R}_0 = \bigcap_{(\gamma,\delta)} \{Z : \| \gamma Z + \delta \| \geqq 1\}, \tag{2.10}$$

where γ, δ range over $\boldsymbol{O}$ but $\gamma \neq 0$. A fundamental domain for $\boldsymbol{H}$ can be shown to be the intersection

$$\boldsymbol{R} = \boldsymbol{R}_0 \bigcap \boldsymbol{R}_\infty \tag{2.11}$$

(with boundary points identified according to $\boldsymbol{H}$ this time). It can be further shown that for $(Z, Z') \in \boldsymbol{R}_0$,

$$\min \boldsymbol{h}(Z, Z') = h_0(k) > c^2/(2k), \tag{2.12}$$

and it is actually achieved at some *low point* (Z_0, Z_0') of R (see [2, §4]). The points of $\boldsymbol{R}$ which lie on some hypersurface $\| \gamma Z + \delta \| = 1$ are said to constitute the *floor* of $\boldsymbol{R}$. If for all points of the floor we can associate a (γ, δ) with $\gamma = 1$, we say that the floor is *simple*, in effect this means the floor consists essentially of $\| Z \| = 1$ (and its "translates" $Z \to Z + \delta$).

For a point on the floor Z, we have $\| \gamma Z + \delta \| = 1$ for some (γ, δ) but $\|\gamma Z + \delta \| > 1$ for all others (i.e., we never have $\| \gamma Z + \delta \| < 1$). In many cases the decimal accuracy in distinguishing the inequality will be very critical.

3. Description of Program. The floor of the fundamental domain is a three-dimensional set in four-dimensional space. Therefore, if we fix any point (X, X') or (R_1, R_2) in the parallelogram (2.9ab) the floor would determine a curve $\boldsymbol{C}$ in (Y, Y') space.

To parametrize $\boldsymbol{C}$ it is more convenient to introduce new coordinates S_1, S_2 analogous with R_1, R_2 as follows:

$$S_1 = k^{1/2}(Y' - Y)/(Y' + Y) \tag{3.1}$$

and, because of our interest in the height $\boldsymbol{h}(Z, Z')$,

$$S_2 = YY'. \tag{3.2}$$

The advantage in using S_1 consists in the fact that the wedge (2.8) becomes transformed into the parallel strip:

$$-h \leqq S_1 \leqq h \tag{3.3}$$

where (recalling $\epsilon_+' = \epsilon_+^{-1}$) we see the bounds are rational:

$$h = k^{1/2}(\epsilon_+ - 1)/(\epsilon_+ + 1) \ (>0). \tag{3.4}$$

Values of h are shown in Table 1 (see below; ΔS_1 is presently explained in (3.14)).

The curve $\boldsymbol{C}$ in the floor lying over each (X, X') or (R_1, R_2) can be parametrized by the ratio Y'/Y or by S_1 in the range (3.3). To see this, define

$$\Phi_0(Z, Z') = \inf_{(\gamma,\delta)} \| \gamma Z + \delta \|, \qquad (0 \neq \gamma \in \boldsymbol{O}, \delta \in \boldsymbol{O}). \tag{3.5}$$

TABLE 1

Range of S_1

k	ϵ_0	ϵ_+	h	ΔS_1
2	$1 + 2^{1/2}$	ϵ_0^2	1	.2
3	$2 + 3^{1/2}$	ϵ_0	1	.2
6	$5 + 2 \cdot 6^{1/2}$	ϵ_0	2	.1
7	$8 + 3 \cdot 7^{1/2}$	ϵ_0	$\frac{7}{3}$	.333···
11	$10 + 3 \cdot 11^{1/2}$	ϵ_0	3	.3
14	$15 + 4 \cdot 14^{1/2}$	ϵ_0	$\frac{7}{2}$	.35
5	$\frac{1}{2}(1 + 5^{1/2})$	ϵ_0^2	1	.2
13	$\frac{1}{2}(3 + 13^{1/2})$	ϵ_0^2	3	.3
17	$4 + 17^{1/2}$	ϵ_0^2	4	.5
21	$\frac{1}{2}(5 + 21^{1/2})$	ϵ_0	3	.3
29	$\frac{1}{2}(5 + 29^{1/2})$	ϵ_0^2	5	.5
33	$23 + 4 \cdot 33^{1/2}$	ϵ_0	$\frac{11}{2}$	.55

Clearly Φ_0 increases monotonically in G where $Z = X + GYi$, $Z' = X' + GY'i$. Thus for each fixed S_1 there is precisely one S_2 for which $\Phi_0 = 1$.

Actually these functions are computationally quite reasonable. We note the inequalities:

$$\| \gamma Z + \delta \| \geqq N(\gamma)^2 N(Y)^2 \tag{3.6}$$

and

$$\| \gamma Z + \delta \| > (\gamma' Y')^2 (\gamma X + \delta)^2, \tag{3.7a}$$

$$\| \gamma Z + \delta \| > (\gamma Y)^2 (\gamma' X' + \delta')^2. \tag{3.7b}$$

Hence $\| \gamma Z + \delta \| > 1$ *unless* (by (2.12))

$$0 < | N(\gamma) | < 2k/c^2 \qquad (\boldsymbol{M}) \tag{3.8}$$

and

$$\begin{aligned} - \gamma X - | \gamma' Y' |^{-1} &< \delta < - \gamma X + | \gamma' Y' |^{-1}, &\qquad (\boldsymbol{M}) \\ - \gamma' X' - | \gamma Y |^{-1} &< \delta' < - \gamma' X' + | \gamma Y |^{-1}. &\qquad (\boldsymbol{M}) \end{aligned} \tag{3.9}$$

Only a finite set of "input" γ satisfies (3.8) (see Table 3 below) if we ignore associates of any γ (values of γ differing by a unit factor). For each γ a finite set of δ and δ' are determined by (3.9) (see Remark (a) in §8 below). The set of admissible γ was worked out beforehand by hand, using factorization laws for the field $\boldsymbol{K}$. These values are stored in the memory as couples (g_1, g_2) where $\gamma = g_1 + g_2\omega$. For any given Z it is no difficult task to program a computer to run through all γ and δ satisfying (3.8) and (3.9). Call this set $\boldsymbol{M}$. Then for practical purposes we are dealing, not with Φ_0 of (3.5) above but with the *numerically equal function* Φ_1 (written in R_1, R_2, S_1, S_2 instead of Z, Z' for convenience),

$$\Phi_1(R_1, R_2, S_1, S_2) = \min \| \gamma Z + \delta \|, \qquad ((\gamma, \delta) \in \boldsymbol{M}). \tag{3.10}$$

Incidentally the fact that $\boldsymbol{M}$ is a finite set makes the "inf" of (3.5) an actual minimum. (It also assures us that infinitely many different boundary surfaces "$\| \gamma Z + \delta \| = 1$" will not accumulate near a point Z, Z' as long as $Y > 0$, $Y' > 0$.) *The machine does keep track, additionally, of the last* (γ, δ) *which produces the minimum in* (3.10), *for later purposes of output.*

We can finally calculate the curve $\boldsymbol{C}$ lying over (R_1, R_2) (or (X, X')) by solving for S_2 so that the floor is now given by one "finitary" equation:

$$\Phi_1(R_1, R_2, S_1, S_2) = 1. \tag{3.11}$$

By virtue of the earlier remarks (see (2.12)) there is a unique solution

$$2/(c^2k) < S_2 \leqq 1 \tag{3.12}$$

and this value of S_2 can be found by a bisection process on S_2 in the interval (3.12). designed to cut off when $| \Phi_1 - 1 | < E$ a preassigned value (see Remark (b) of §8 below). As a check on accuracy, the machine prints out the difference "error S_2" between the last two approximations to S_2 (by bisection). In practice with $E = .001$, usually "error S_2" $< .01$.

Now consider the inverse of (3.11) as $S_2 = f(R_1, R_2, S_1)$. For this function, it is easily seen that if $k \not\equiv 1 \pmod 4$ there is complete "$\pm R_1$ and $\pm R_2$ symmetry." This follows from the fact that we can change X and X' to $-X$ and $-X'$ while we can interchange X and Y with X' and Y'. (If $k \equiv 1 \pmod 4$ the symmetry is not reflected in R_1 and R_2 both, e.g., only "symmetry about the origin" can be used.) Thus we search the following parallelopiped:

$$\begin{aligned} -.5 \leqq R_1 \leqq 0 \quad &\text{in } N_1 \text{ steps of} \quad \Delta R_1 \ (\Delta R_1 = .1, N_1 = 6), \\ -.5 \leqq R_2 \leqq 0 \quad &\text{in } N_2 \text{ steps of} \quad \Delta R_2 \ (\Delta R_2 = .1, N_2 = 6), \end{aligned} \tag{3.13}$$

$$-h \leqq S_1 \leqq h \quad \text{in } N_3 \text{ steps of} \quad \Delta S_1 \quad \text{(Table 1)}. \tag{3.14}$$

(We use $-.5 \leqq R_2 \leqq .5$ if $k \equiv 1 \bmod 4$ since we lack one of the symmetries.) Here ΔR_1, ΔR_2, ΔS_1 are part of the input data as well as the numbers N_1, N_2, N_3 of steps (and of course the starting values $-.5$, $-.5$ and $-h$).

The output represents essentially the floor as determined by inverting (3.11) to obtain

$$S_2 = f(R_1, R_2, S_1) \ ; \tag{3.15}$$

it consists of the sequence of $N_1N_2N_3$ "points" in 13 columns:

$$R_1, R_2, S_1, S_2, \qquad X, X' \qquad Y, Y', \qquad g_1, g_2, d_1, d_2, \qquad (\text{error } S_2) \tag{3.16}$$

given in decimal form (to four places). By scanning the data one could easily spot low points as small values of S_2. The values of $\gamma = g_1 + \omega g_2$ and $\delta = d_1 + \omega d_2$ are printed to correspond to the *last* value of $(\gamma, \delta) \in \boldsymbol{M}$ arising (in finding S_2) for which Φ_1 takes its minimum. This way, if we are dealing with a nonsimple floor we should find $\gamma \neq 1$ or $(g_1, g_2) \neq (1, 0)$. More will be said about the output of (γ, δ) later on (see Table 3 below).

4. Study of Case $k = 6$. To illustrate the main run for a typical case take $k = 6$. Here the function of (3.14), $S_2 = f(R_1, R_2, S_1)$ is calculated for the following $1476 = 6 \cdot 6 \cdot 41$ points

$$\begin{aligned} -.5 &\leqq R_1 \leqq 0, & \Delta R_1 &= .1 & &(6 \text{ values}), \\ -.5 &\leqq R_2 \leqq 0, & \Delta R_2 &= .1 & &(6 \text{ values}), \\ -2 &\leqq S_1 \leqq 2, & \Delta S_1 &= .1 & &(41 \text{ values}). \end{aligned}$$

Let us concentrate on each curve $\boldsymbol{C}$ which lies over (R_1, R_2). It is represented by 41 points (S_1, S_2) which in turn lie on a (variable) number of hypersurfaces "$\| \gamma Z + \delta \| = 1$". For example, when $R_1 = -.3$ and $R_2 = -.4$, the *print-outs reveal* that $\boldsymbol{C}$ consists of five analytic pieces lying on the following hypersurfaces:

$$\begin{aligned} \| Z - 1 \| &= 1 \quad \text{for} \quad -2.0 \leqq S_1 \leqq -1.7, \\ \| (3 + 6^{1/2})Z + (2 + 6^{1/2}) \| &= 1 \quad \text{for} \quad -1.6 \leqq S_1 \leqq -1.0, \\ \| (2 + 6^{1/2})Z + (3 + 6^{1/2}) \| &= 1 \quad \text{for} \quad -.9 \leqq S_1 \leqq 1.0, \\ \| Z + 1 \| &= 1 \quad \text{for} \quad 1.1 \leqq S_1 \leqq 1.9, \\ \| Z - 1 + 6^{1/2} \| &= 1 \quad \text{for} \quad 2.0 = S_1 . \end{aligned}$$

The values of S_2 vary, *generally* displaying several relative minima for the range

−.5		−.4		−.3		−.2		−.1		0	$= R_1$ / $R_2 =$
A * .33		A * .47		A * .64		A * .81		A * .95		A * 1.00	0
	A * .40		A * .38		A * .52		A * .70		A * .86		
A * .69		A * .50		A * .33		A * .41		A * .57		A * .74	−.1
	A * .68		A * .61		A * .41		A * .41		A * .45		
A * .62		A * .63		A * .61		A * .51		A * .31		A * .41	−.2
	A * .56		A * .55		A * .49		A * .45		A * .35		
A * .45		A * .46		A * .45		A * .36		A * .28		A * .40	−.3
	A * .34		A * .33		AB * .27		ABC * .31		AB * .34		
AB * .25		AB * .20		ABC * .26		AB * .32		AB * .39		AB * .43	−.4
	ABD * .24		ABC * .26		AB * .36		AB * .42		AB * .46		
ABD * .25		AB * .29		AB * .37		AB * .44		AB * .48		AB * .50	−.5

FIG. 1.—Value of min S_2 as S_1 varies for fixed values of R_1 and R_2 (between $-.5$ and 0) on the lower left-hand quadrant.

of S_1; indeed S_2 increases from .260 at $S_1 = -2$ to .529 at $S_1 = 1.7$ finally decreasing to .499 at $S_1 = 2$ (ignoring minor fluctuations). In Figure 1 we indicate the minimum (computed) value of S_2 over the range $-2 \leqq S_1 \leqq 2$ as a function of R_1 and R_2 alone, i.e., we indicate the minimum value of S_2 on $\boldsymbol{C}$ for each of 36 (R_1, R_2). In order to give a better picture of how this minimum varies, we also calculate the values at the 25 mid-points of the squares by a supplementary computation.

Moreover, to obtain some idea of how complicated the "piecing" of hypersurfaces becomes, in Figure 1, we denote by

A the presence of $| N(\gamma) | = 1$,

B the presence of $| N(\gamma) | = 2$,

C the presence of $| N(\gamma) | = 3$,

D the presence of $| N(\gamma) | = 4$

among the hypersurfaces $\| \gamma Z + \delta \| = 1$ occurring over (R_1, R_2). Thus the last point, $R_1 = -.3$, $R_2 = -.4$, bears the information "*ABC*, .26", etc. We see that "*D*" occurs only twice, in the lower left-hand corner. Clearly, we can fail to find hypersurfaces by round-off error as well as by using a grid which is not sufficiently fine, so that we must regard results of this nature as tentative and unreliable indications of a minimum degree of complication. (Of course, several different δ can occur with each γ but they are not distinguished here.)

Finally, the low-point suggested by Figure 1 is

$$R_1 = -.4, \qquad R_2 = -.4, \qquad S_1 = -1, \qquad S_2 = .2 \tag{4.1}$$

or, in the original coordinate,

$$\begin{aligned} Z &= -\frac{2}{5}(1 + 6^{1/2}) + \frac{i}{5}(1 + 6^{1/2}), \\ Z' &= -\frac{2}{5}(1 - 6^{1/2}) + \frac{i}{5}(-1 + 6^{1/2}). \end{aligned} \tag{4.2}$$

Subsequent exploration with a finer grid seems to justify the conjecture that (4.1) or (4.2) is the low-point (see Remark (c) in §8 below).

5. Nonsimplicity Theorem. We now examine the output data to see where $\gamma \neq 1$. This is a necessary condition for a nonsimple floor (but not a sufficient one since we can have the coincidental occurrence $\| \gamma Z + \delta \| = \| Z + \delta_1 \| = 1$, i.e., the boundary point Z can lie on two boundary surfaces at once).

Our attention is immediately drawn to the special values $R_1 = 0$, $R_2 = -1/2$, for which the (floor) output shows $\gamma \neq 1$ (except when $k = 3$ or 5). The values of S_1 and S_2 are not always the same but for simplicity we consider the point Z_* corresponding to $S_1 = 0$ and $S_2 = 1/4$ or

$$Z_* = \tfrac{1}{2}(-\omega + i), \qquad Z'_* = \tfrac{1}{2}(-\omega' + i). \tag{5.1}$$

Now Z_* does not generally lie on the floor, but still

$$\| 2Z_* + \omega \| = 1. \tag{5.2}$$

We shall soon try to see if for *all* $\delta \in \boldsymbol{O}$ the following holds for $Z = Z_*$:

$$\| Z + \delta \| > 1. \tag{5.3}$$

From (5.2) it follows that in (3.11) $\Phi_1(0, -1/2, 0, 1/4) \leqq 1$. Thus if S_2' is the root of $\Phi_1(0, -1/2, 0, S_2') = 1$, it follows that $S_2' \geqq 1/4$ and (5.3) still holds for the point of the floor corresponding to this root. Hence, (see [2, §8]) there is no hypersurface of type (5.3) on the floor at $R_1 = 0, R_2 = -1/2, S_1 = 0, S_2' (\geqq 1/4)$.

Actually to demonstrate (5.3) for $Z = Z_*$, consider its negation, $16 \| Z_* + \delta \| \leqq 16$. This can be expanded as

$$(\sigma^2 + 1)(\sigma'^2 + 1) \leqq 16 \tag{5.4}$$

where $\sigma = 2\delta - \omega$, which is irrational when δ varies in $\boldsymbol{O}$. Clearly (5.4) provides a bound on $|\sigma|$ and $|\sigma'|$ hence only a finite number of quadratic *fields* can permit the inequality (5.4). A quick check reveals the only counterexamples satisfying (5.4) and $\sigma \equiv \omega \pmod 2$ are

$$\begin{aligned} \sigma = \pm\left(\frac{1 + 5^{1/2}}{2}\right), \pm\left(\frac{-3 + 5^{1/2}}{2}\right)&, \\ \pm 2^{1/2}, \pm 3^{1/2}, \pm 2 \pm 3^{1/2}, \pm\left(\frac{-3 + 13^{1/2}}{2}\right)&. \end{aligned} \tag{5.5}$$

Therefore, the only cases to check are $k = 5, 2, 3, 13$. Actually $k = 5$ is easily disposed of as *simple* from (3.8) since in $\boldsymbol{K}$, $|N(\gamma)| = 1$ or $\geqq 4$. To dispose of $k = 2, 3$ and $k = 13$ we search the outputs again and we obtain the following counterexamples by trial and error:

$k = 2$. Let

$$Z_* = -\frac{1}{2} \cdot 2^{1/2} + i, \qquad Z_*' = \frac{1}{2} \cdot 2^{1/2} + \frac{i}{2}.$$

Thus $\| 2^{1/2} Z_* + 1 \| = 1$ while if $\delta = m + n2^{1/2}$, then

$$\| Z_* + \delta \| = \{(m + [n - \tfrac{1}{2}]2^{1/2})^2 + 1\}\{(m - [n - \tfrac{1}{2}]2^{1/2})^2 + \tfrac{1}{4}\}.$$

But $\| Z_* + \delta \| \leqq 1$ only if $|m^2 - 2(n - \tfrac{1}{2})^2| < 1$ or if $(2n - 1) + m2^{1/2}$ is a unit. Then trying units, we find $\| Z_* + \delta \| \geqq 9/8 > 1$ for $\delta \in \boldsymbol{O}$.

$k = 3$. Let $Z_* = (-4 - 5 \cdot 3^{1/2} + 7i)/10$, $Z_*' = (-4 + 5 \cdot 3^{1/2} + 7i)/10$ (not an easy point to conjecture but it appeared on the print-outs with $\gamma = 1 + 3^{1/2}$, $\delta = 2 + 3^{1/2}$)! Here

$$\| (1 + 3^{1/2})Z_* + (2 + 3^{1/2}) \| = \| (1 + 3^{1/2})(1 + 7i)/10 \| = 1.$$

Meanwhile, after somewhat more labored calculation we verify

$$\| Z_* + \delta \| \geqq \tfrac{37}{25} > 1 \text{ for } \delta \in \boldsymbol{O}.$$

$k = 13$. This case resembles $k = 2$ (see Remark (d) of §8 below). Let

$$Z_* = -\frac{1}{2}\omega + i, \qquad Z_*' = -\frac{1}{2}\omega' + \frac{i}{4}, \qquad \left(\omega = \frac{1}{2}[1 + 13^{1/2}]\right).$$

Thus $\| 2Z_* + \omega \| = 1$ while with $\delta = m + n\omega$ we note that $\| Z + \delta \| > 1$ unless, with $\lambda = 2m + (2n - 1)\omega$, $| N(\lambda) | \leqq 4$. But if $N(\lambda) = \pm 1, \pm 3$, since $\| Z + \delta \| > (\lambda/8)^2 + (\lambda'/2)^2$ we reduce our choice (by $\max | \lambda |, | \lambda' | < 8$) to $\lambda = \pm(3 - 13^{1/2})/2, \pm(1 + 13^{1/2})/2, \pm(5 + 13^{1/2})/2$; (recall $\lambda \equiv \omega \bmod 2$). By a close margin, again,

$$\| Z_* + \delta \| \geqq (806 - 150\sqrt{13})/256 = 1.03\cdots > 1(!).$$

MAIN THEOREM. *The only quadratic field of class number unity with a simple floor for the Hilbert fundamental domain is the field of* $5^{1/2}$.

In completing this proof, *we did not depend on the print-outs*, but we verified all inequalities rigorously. In what follows we shall depend on the print-outs because of the amount of labor involved in verifying even a single (print-out) S_2.

6. Low-Points. We first inspect the print-outs in order to conjecture the minimum value of S_2. (A more difficult job, of course, is to invent some justification for the conjectured minimum.) In certain cases, $k = 2, 3, 5, 6, 13$ a reasonable conjecture could be made from this and earlier work ([2], [3]). In other cases, $k = 11, 21, 29, 33$ no "exact" conjecture seems available so we content ourselves with repeating the decimal values in the output. In still other cases, however, $k = 7, 14, 17$, we can "explain" the minimum output but without sufficient conviction to justify a conjecture that this value is the low-point.

A frequent situation for some K is where the output produces $S_1 = 0$, or $Y = Y' (= Y_k)$. Then the *precise* value of Y_k is determined by the condition

$$\| \gamma Z + \delta \| = [(\gamma X + \delta)^2 + \gamma^2 Y_k^2][(\gamma' X' + \delta')^2 + \gamma'^2 Y_k^2] = 1 \tag{6.1}$$

drawn from the output values of γ and δ. Then $S_2 = Y_k^2$. This happens for $k = 2, 3, 7, 13, 14, 17$.

The relevant information is collected in final form in Table 2. In each case, the

TABLE 2

Possible Low-Points

k $(c = 1)$[1]	Z, Z'	(γ, δ)	$h(Z, Z')$[2]
2	$-\frac{1}{2}(1 + 2^{1/2}) + iY_2, \quad -\frac{1}{2}(1 - 2^{1/2}) + iY_2$	$(1, \omega)$	$.475\cdots$
3	$\frac{1}{2}(-3^{1/2} + i), \quad \frac{1}{2}(3^{1/2} + i)$	$(1, 0)$	$.25$
6	$\frac{1}{5}(1 + 6^{1/2})(-2 + i), \quad \frac{1}{5}(1 - 6^{1/2})(-2 - i)$	$(1, 0)$	$.20$
7	$-\frac{2}{5}\cdot 7^{1/2} + iY_7, \quad \frac{2}{5}\cdot 7^{1/2} + iY_7$	$(3 + \omega, 2 + \omega)$	$.245\cdots$
11	$-1.5266 + i\,.0804, \quad 1.1266 + i\,1.6034$	$(2 - \omega, -2)$	$.129\cdots$
14	$-\frac{1}{2} - \frac{3}{10}\cdot 14^{1/2} + \frac{3}{10}i, \quad -\frac{1}{2} + \frac{3}{10}\cdot 14^{1/2} + \frac{3}{10}i$	$(4 + \omega, 11 + 3\omega)$	$.09$
$(c = 2)$[2]			
5	$\frac{1}{2}(-\omega + i[-5^{1/2}\omega']^{1/2}), \quad \frac{1}{2}(-\omega' + i[5^{1/2}\omega]^{1/2})$	$(1, 0)$	$.559\cdots$
13	$-\frac{1}{2}\omega + i/2, \quad -\frac{1}{2}\omega' + i/2$	$(2, \omega)$	$.25$
17	$-\frac{3}{10} - \frac{4}{10}\omega + iY_{17}, \quad -\frac{3}{10} - \frac{4}{10}\omega' + iY_{17}$	$(2 - \omega, -\omega)$	$.405\cdots$
21	$-1.6956 + i\,.2014, \quad .5956 + i\,.8639$	$(\omega, 2 + \omega)$	$.174\cdots$
29	$-1.5232 + i\,2.2059, \quad .5232 + i\,.0818$	$(2, 1 + \omega)$	$.180\cdots$
33	$.0372 + i\,1.8059, \quad -.5372 + i\,.1342$	$(1, 0)$	$.242\cdots$

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[1] Recall $\omega = k^{1/2}$ and $\omega' = -k^{1/2}$ for $c = 1$ and $\omega = (1 + k^{1/2})/2$, $\omega' = (1 - k^{1/2})/2$ for $c = 2$.

[2] For $k = 2, 7, 17$, $h = Y_k^2$ where $Y_2^2 = (2\cdot 6^{1/2} - 3)/4$, $Y_7^2 = (10\cdot 2^{1/2} - 8)/25$, $Y_{17}^2 = (5\cdot 417^{1/2} - 21)/200$.

output values of γ and δ are listed in Table 2 for the reader's convenience in reconstructing the optimum, (but naturally, several distinct (γ, δ) may be equally capable of producing $\| \gamma Z + \delta \| = 1$).

7. Bounding Surfaces. We now return to the question of how complicated the floor of the fundamental domain $\boldsymbol{R}$ really is. In every sense, we are establishing a *minimum level of complexity*, since our grid may not be fine enough to catch all hypersurfaces which make up the floor.

Let us look at the floor, however, in terms of *zones*. Let us define $\boldsymbol{Z}_{\gamma,\delta}$ (for $\gamma \neq 0$) the *(γ, δ)-zone* as the set of (Z, Z') in $\boldsymbol{U} \times \boldsymbol{U}$ for which

$$\| \gamma Z + \delta \| \leqq \| \gamma (Z + \rho) + \delta \| \tag{7.1}$$

for every $\rho \in \boldsymbol{O}$. Then the portion of the floor lying on $\| \gamma Z + \delta \| = 1$ can be translated (by ρ) so as to lie on $\boldsymbol{Z}_{\gamma,\delta}$. In a sense, then, the simplest possible description of the floor would consist of the enumeration of the values of the residue classes $\delta \pmod{\gamma}$. The translations (by ρ) involve an additional consideration of complexity which we thereby ignore.

We list the values of $\delta \pmod{\gamma}$ in Table 3 omitting the case $\gamma = 1$ $(\delta = 0)$, which always occurs (since the point $Z = i$, $Z' = i$ must necessarily lie on the floor

TABLE 3

Analytic Segments of the Floor: $\lvert \gamma Z + \delta \rvert^2 \lvert \gamma' Z' + \delta' \rvert^2 = 1$

k $(c = 1)$	Input	Output	
	γ (where $\lvert N(\gamma) \rvert < 2k/c^2$)	$\delta \pmod{\gamma}$ (excl. $\gamma = 1$)	max $\lvert N(\gamma) \rvert$
2	$1, 2^{1/2}$	$1 \pmod{2^{1/2}}$	2
3	$1, 1 + 3^{1/2}, 3^{1/2}, 2$	$1 \pmod{1 + 3^{1/2}}$	2
6	$1, 2 + 6^{1/2}, 3 + 6^{1/2}, 2, 1 \pm 6^{1/2}, 6^{1/2}, 4 + 2 \cdot 6^{1/2}, 3, 4 \pm 6^{1/2}$	$1 \pmod{2 + 6^{1/2}}$; $\pm 1 \pmod{3 + 6^{1/2}}$; $1 + 6^{1/2} \pmod{2}$	4
7	$1, 3 + 7^{1/2}, 2 \pm 7^{1/2}, 2, 1 \pm 7^{1/2}, 7^{1/2}, 6 + 2 \cdot 7^{1/2}, 3, 4 \pm 7^{1/2}, 4 \pm 2 \cdot 7^{1/2}$	$1 \pmod{3 + 7^{1/2}}$; $\pm 1 \pmod{2 \pm 7^{1/2}}$; $1, 7^{1/2} \pmod{2}$	4
11	$1, 3 + 11^{1/2}, 2, 4 \pm 11^{1/2}, 2 \pm 11^{1/2}, 6 + 2 \cdot 11^{1/2}, 3, 11^{1/2}, 4, 9 + 3 \cdot 11^{1/2}$	$1 \pmod{3 + 11^{1/2}}$; $1, 14^{1/2} \pmod{2}$; $\pm 1, \pm 2 \pmod{4 \pm 11^{1/2}}$; $\pm 2, \pm 3 \pmod{2 \pm 11^{1/2}}$	7
14	$1, 4 + 14^{1/2}, 2, 3 \pm 14^{1/2}, 7 + 2 \cdot 14^{1/2}, 8 + 2 \cdot 14^{1/2}, 3, 2 \pm 14^{1/2}, 5 \pm 14^{1/2}, 1 \pm 14^{1/2}, 14^{1/2}, 4, 12 + 3 \cdot 14^{1/2}, 6 \pm 2 \cdot 14^{1/2}$	$1 \pmod{4 + 14^{1/2}}$; $1, 1 + 14^{1/2} \pmod{2}$; $\pm 2 \pmod{3 \pm 14^{1/2}}$; $\pm 3 \pmod{2 \pm 14^{1/2}}$	10
$(c = 2)$			
5	1	. . .	1
13	$1, \frac{1}{2}[1 \pm 13^{1/2}], 2$	$\frac{1}{2}[1 \pm 13^{1/2}] \pmod{2}$; $\pm 1 \pmod{\frac{1}{2}[1 \pm 13^{1/2}]}$	3
17	$1, \frac{1}{2}[3 \pm 17^{1/2}], 2, \frac{1}{2}[1 \pm 17^{1/2}], 3 \pm 17^{1/2}, \frac{1}{2}[7 \pm 17^{1/2}]$	$1 \pmod{\frac{1}{2}[3 \pm 17^{1/2}]}$	2
21	$1, \frac{1}{2}[3 + 21^{1/2}], 2, \frac{1}{2}[1 \pm 21^{1/2}], \frac{1}{2}[7 + 21^{1/2}], 3$	$\pm 1 \pmod{\frac{1}{2}[3 + 21^{1/2}]}$; $\frac{1}{2}[1 \pm 21^{1/2}] \pmod{2}$; $\pm 2 \pmod{\frac{1}{2}[1 \pm 21^{1/2}]}$	5
29	$1, 2, \frac{1}{2}[3 \pm 29^{1/2}], \frac{1}{2}[1 \pm 29^{1/2}], 3, \frac{1}{2}[9 \pm 29^{1/2}]$	$1, \frac{1}{2}[1 \pm 29^{1/2}] \pmod{2}$; $\pm 2 \pmod{\frac{1}{2}[3 \pm 29^{1/2}]}$	5
33	$1, \frac{1}{2}[5 \pm 33^{1/2}], 6 + 33^{1/2}, \frac{1}{2}[7 \pm 33^{1/2}], 2, \frac{1}{2}[3 \pm 33^{1/2}], \frac{1}{2}[1 \pm 33^{1/2}], 5 \pm 33^{1/2}, 3, 11 + 2 \cdot 33^{1/2}, 12 + 2 \cdot 33^{1/2}, \frac{1}{2}[9 \pm 33^{1/2}], 4, 7 \pm 33^{1/2}, \frac{1}{2}[41 \pm 7 \cdot 33^{1/2}]$	$1 \pmod{\frac{1}{2}[5 \pm 33^{1/2}]}$	2

only on hypersurface $\| Z \| = 1$). In several cases the values of δ (mod γ) were also drawn from supplementary runs which were made to explore the neighborhoods of the low-points in Table 2.

We do take symmetries ($X \leftrightarrow -X$, $X' \leftrightarrow -X'$) and ($Z \leftrightarrow Z'$) into account, so that for every δ (mod γ) occurring, $-\delta$ (mod γ) and $\pm\delta'$ (mod γ') are also present; they are listed separately insofar as $\delta \not\equiv -\delta$ (mod γ) and γ and γ' are nonassociated, etc.

There are 12 "pieces" present (and translates) e.g., for $k = 14$ (i.e., 12 residue classes δ mod γ including $\gamma = 1$). There is no reason to be sure that any listing is complete but we might feel relatively sure of the smaller k. At any rate, an actual count of output pairs (γ, δ) for $k = 5$ yields 7 different pairs (γ, δ) for the *simple* floor; namely, $\gamma = 1$ and $\delta = 0, \pm 1, \frac{1}{2}(\pm 1 \pm 5^{1/2})$. By contrast, an actual count of output values for $k = 14$ yields 51 pairs (γ, δ) belonging to the 12 "pieces" (and 39 translates).

8. Concluding Remarks. (a) In calculating δ and δ' from (3.9), the procedure used is to note that if $\delta = d_1 + d_2\omega$ and $\delta' = d_1 + d_2\omega'$ then $d_1 = \psi_2(\delta, \delta')$ and $d_2 = \psi_2(\delta, \delta')$ by just solving linear equations. Hence, if $\delta_1 < \delta < \delta_2$ and $\delta_1' < \delta' < \delta_2'$, it is clear that

$$(8.1) \qquad \psi_1(\delta_1, \delta_1') < d_1 < \psi_1(\delta_2, \delta_2'), \qquad \psi_2(\delta_1, \delta_2') < d_2 < \psi_2(\delta_2, \delta_1').$$

Thus to explore the range (3.9) it is only necessary to explore the range (8.1) for integers d_1 and d_2.

(b) To avoid losing a value of d_1 or d_2 by round-off errors, the computer artificially rounds up (or down) by actually using not (8.1), but

$$(8.2) \qquad -.01 + \psi_1(\delta_1, \delta_1') \leqq d_1 \leqq +.01 + \psi_1(\delta_2, \delta_2'), \text{ etc.}$$

Likewise, equation (3.12) is interpreted as $2/c^2k^2 \leqq S_2 \leqq 1.1$, partly to make allowance for the cases ($R_1 = 0, R_2 = 0$) where $S_2 = 1$ theoretically.

(c) We are constantly being reminded of our dependence on the "decimal world". By an undeserved stroke of luck, the conjectured low point for $k = 6$ (see Table 2) appears as the print-out

$$R_1 = -.4, \qquad R_2 = -.4, \qquad S_1 = -1.0, \qquad S_2 = .2005 \text{ (calc.)}.$$

Here R_1, R_2, S_1 falls in our scanning range (3.13) since ΔR_1, ΔR_2, and ΔS_1 were taken as .1 for lack of any more inspired choice. Likewise the counterexample (in §5) for the nonsimplicity proof for $k = 3$ is a fortunate result of "good decimal" choices for ΔR_1, ΔR_2, and ΔS_1.

(d) In the counterexamples of §5 for $k = 2$ (or $k = 13$) the printouts suggest that $Z = -\frac{1}{2}\omega + iY$ and $Z = -\frac{1}{2}\omega' + iY'$ will do for a range or values of Y and Y' *not too far from the ones attempted*; e.g., $Y = 1$ and $Y' = \frac{1}{2}$ (or $\frac{1}{4}$ as long) as $YY' = \frac{1}{2}$ (or $\frac{1}{4}$). The printouts do not suggest a reason for any freedom for Y and Y' in the choice of counterexamples.

The actual computing time was roughly 250 points/minute and this includes, on the average, 6 bisections for the location of S_2. The running time seemed largely independent of the *number* of input values γ (which varied from 1 when $k = 5$ to

23 when $k = 33$). This may be partly due to the fact that the high internal speed makes the output (to tape) a lengthly process causing some stability (in the ratio of 250 points/minute). Meanwhile, for many input values of γ, the inequalities (3.9) provide no value of δ to test. The time for the total computation was roughly one hour spread over several weeks in July and September, 1964.

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