A Graph Technique for Inverting Certain Matrices

By B. H. Mayoh

1. Introduction. Several authors ([5], [6], [7], [8, pp. 188–195], [9, pp. 112–118], [10], and [11]) have described a method for inverting a matrix M, in which M is partitioned, certain smaller submatrices are inverted, and these inverses are combined by the Frobenius-Schur relation [8, p. 189]. Though the results of E. Bodewig in [7] do not justify his assertion that this is the most efficient general way of inverting matrices (see [12, pp. 125–130]) the method is valuable if the smaller submatrices can be easily inverted.

In this paper two graphs are associated with a matrix M, and used to permute the rows and columns of M until it or its transpose are in the form of Figure 1. On matrices of this form the above inversion method is very effective.

A quite different graph theoretical approach has been described by F. Harary [1]. There the matrix M to be inverted is permuted into block triangular form but under the very severe restriction that the same permutation is applied to the rows and the columns.

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2. Single Bordered Block Diagonal Forms.

Definition. A partitioned $n \times n$ matrix M is said to be in *singly bordered block diagonal form* (b.b.d.f.) if either it or its transpose M' are of the form shown in Figure 1.

THEOREM. *Suppose that an $n \times n$ matrix M is in b.b.d.f. Then M is nonsingular if and only if $D_1 \cdots D_m$, F are square nonsingular matrices.*

Proof. This follows immediately from the Frobenius-Schur relation and the fact that a matrix in block diagonal form is nonsingular if and only if each submatrix on the diagonal is nonsingular and hence square.

3. The Graphs Associated with a Matrix. The *column graph* G of a matrix M is given by interchanging "row" and "column" throughout the following definition of the *row graph* G_R of M. The vertices of G_R shall be the rows of M, and the edges of G_R are given by:

> there is an edge joining two vertices if and only if they are distinct and there is a column of M with nonzero entries in the two rows corresponding to the vertices.

Clearly the row and column graphs of a matrix M are invariant under any permuta-

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$$\begin{array}{ccccc} D_1 & 0 & 0 & 0 & H_1 \\ 0 & D_2 & 0 & 0 & H_2 \\ 0 & 0 & \cdot & 0 & \cdot \\ 0 & 0 & 0 & D_m & H_m \\ 0 & 0 & \cdot & 0 & F \end{array}$$

FIGURE 1

tion of the rows and columns of M. This is only true for the directed graphs associated with a matrix in ([1], [2], [3]) if the same permutation is applied to both the rows and columns.

By a *component* of an undirected graph G we mean a maximal connected subgraph of G. Each vertex v_i of G lies in just one component; if this component becomes disconnected when v_i and all edges involving v_i are removed, then v_i is said to be a *point of attachment* of G. More generally, if removing a set S of vertices of G and the edges involving them leaves G disconnected, then S is said to be an *attachment set.* In particular, if there are any points of attachment in a graph G, they form an attachment set A, and the components of the graph G^*, remaining when A and the edges involving A are deleted from G, are called the *subcomponents* of G.

Now suppose that S is an attachment set in the row graph of a matrix M. Let $R_1 \cdots R_m$ be the subsets of the set of rows of M that correspond to the m components of the row graph once S and the attendant edges have been deleted. Then each column of M has nonzero entries in rows from at most one R, so one can order the rows and columns of M in such a way that:

(1) A row in R_i comes before any row that corresponds to a suppressed vertex of the row graph;

(2) A column with a nonzero entry in a row of some R_i comes before any column whose only nonzero entries are in rows that correspond to suppressed vertices of the row graph; and for $i < j$;

(3) A row in R_i comes before any row in R_j ;

(4) A column with a nonzero entry in a row of R_i comes before any column with a nonzero entry in a row of R_j .

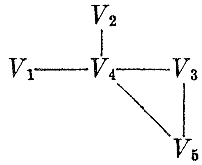
This ordering puts M in b.b.d.f. but the submatrix F may not be square. If there is a diagonal submatrix D_i with more rows than columns, the matrix M has no inverse; if there is a D_i with more columns than rows, it is absorbed into F (i.e., the rows of D_i are added to those corresponding to S and the ordering redefined). Thus we may suppose that F, $D_1 \cdots D_n$ are square, and use the Frobenius-Schur relation to determine the inverse of M.

Dually, the choice of an attachment set in the column graph also allows one to permute M into b.b.d.f. and thereby find its inverse.

Example. Suppose M is the following matrix:

$$\begin{array}{ccccc} 0 & 2 & 0 & 6 & 0 \\ 0 & 0 & 7 & 4 & 6 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 6 & -1 & 5 \\ 1 & 0 & 0 & 3 & 0 \end{array}$$

The column graph of this is:



Noting that V_4 is an attachment vertex, we permute M to:

$$\begin{array}{ccccr} 1 & 0 & 0 & 0 & 3 \\ 0 & 7 & 6 & 0 & -4 \\ 0 & 6 & 5 & 0 & -1 \\ 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{array}$$

This matrix is easily inverted, and appropriate permutation then gives M^{-1}.

4. The Choosing of an Attachment Set. When inverting a matrix by hand, one often has several attachment sets available. Factors influencing the choice between them, are (1) the graph would split into as many components as possible, (2) the attachment set should be small—otherwise many of the D's are lost in making F square—and (3) the diagonal submatrices are of roughly the same size.

When working on a computer, the algorithm in [4] is used to determine the points of attachment and subcomponents of the row and column graphs of the matrix M. The only choice is as to which of the two graphs should be ignored.

In either case one should use both the row and the column graph of M since they may be of quite different degrees of complexity. Furthermore, one should consider the possibility of using the methods of this paper iteratively so that the diagonal submatrices $D_1, \cdots, D_n, F$ are also put in their most easily invertible form.

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