Evaluation at Half Periods of Weierstrass' Elliptic Functions with Double Periods 1 and $e^{i\alpha}$

By Chih-Bing Ling

In two previous papers [1], [2], the following three Weierstrass' elliptic functions at half periods were evaluated to 16D for the cases of rectangular and rhombic primitive period-parallelograms with double periods $2\omega_1 = 1$ and $2\omega_2 = ai$ and $\frac{1}{2} + ci$, respectively:

$$e_1 = \wp(\omega_1), \qquad e_2 = \wp(\omega_2), \qquad e_3 = \wp(\omega_3), \tag{1}$$

where ω_3 is defined by

$$\omega_1 + \omega_2 + \omega_3 = 0. \tag{2}$$

In addition, two related coefficients σ_4 and σ_6 were also evaluated to 16D. The coefficient σ_{2s} for $s \geqq 2$ is defined by the double series

$$\sigma_{2s}(2\omega_1\,, 2\omega_2) = \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(2m\omega_1 + 2n\omega_2)^{2s}}, \tag{3}$$

where the accent on the summation sign denotes the omission of simultaneous zeros of m and n. The coefficient σ_{2s} in these two cases has been further evaluated up to $2s = 50$, also to 16D. The values up to $2s = 20$ were tabulated in a recent paper [3].

In this paper, the evaluation is extended to the case in which the double periods are $2\omega_1 = 1$ and $2\omega_2 = e^{i\alpha}$. It is not difficult to show that

$$\sigma_{2s}(1, e^{i\alpha}) = (1 + e^{i\alpha})^{-2s}\sigma_{2s}(1, \tfrac{1}{2} + ci) \qquad (s \geqq 2) \tag{4}$$

and similarly,

$$\begin{aligned} e_1(1, e^{i\alpha}) &= (1 + e^{i\alpha})^{-2}e_3(1, \tfrac{1}{2} + ci), \\ e_2(1, e^{i\alpha}) &= (1 + e^{i\alpha})^{-2}e_2(1, \tfrac{1}{2} + ci), \\ e_3(1, e^{i\alpha}) &= (1 + e^{i\alpha})^{-2}e_1(1, \tfrac{1}{2} + ci), \end{aligned} \tag{5}$$

where

$$c = \frac{1}{2}\tan\frac{\alpha}{2}. \tag{6}$$

It is thus seen that, apart from a complex multiplier, σ_{2s} and e_r can be evaluated by using the same method as in the case of rhombic primitive period-parallelogram.

Furthermore, it can be shown that when α and β are supplementary angles,

$$\sigma_{2s}(1, e^{i\beta}) = e^{2si\alpha}\sigma_{2s}(1, e^{i\alpha}) \qquad (s \geqq 2) \tag{7}$$

and

$$\begin{aligned} e_1(1, e^{i\beta}) &= e^{2i\alpha}e_2(1, e^{i\alpha}), \\ e_2(1, e^{i\beta}) &= e^{2i\alpha}e_1(1, e^{i\alpha}), \\ e_3(1, e^{i\beta}) &= e^{2i\alpha}e_3(1, e^{i\alpha}). \end{aligned} \tag{8}$$

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TABLE 1

$\alpha°$	σ_4	σ_6
10 170	2.20327 53592 69607 (3) ∓8.01926 64866 63102 (2)i	−6.28162 53571 93760 (4) ±3.62669 80909 24197 (4)i
20 160	1.13982 48849 40138 (2) ∓9.56426 64055 99553 (1)i	−5.79789 14112 90271 (2) ±1.00422 42501 12197 (3)i
30 150	1.50455 42089 69734 (1) ∓2.60596 43326 77182 (1)i	0 ±1.06195 78899 46210 (2)i
40 140	1.64345 02330 26281 ∓9.32046 94280 08436 i	1.08183 16750 82049 (1) ±1.87378 74264 79455 (1)i
50 130	−5.29056 91229 52277 (−1) ∓3.00043 08482 43006 i	7.60880 13435 54263 ±4.39294 35039 11440 i
60 120	0	5.86303 16934 25402
70 110	1.37646 54366 09637 ±1.15499 16401 87420 i	3.65983 74207 88683 ∓2.11300 81200 82612 i
80 100	2.64581 45492 33692 ±9.62997 74130 95135 (−1)i	1.14643 33070 49294 ∓1.98568 07352 98588 i
90	3.15121 20021 53898	0

TABLE 2

$\alpha°$	e_1	e_2	e_3
10 170	1.06629 64687 30742 (2) ∓1.88016 41620 31514 (1)i	1.06629 63248 53731 (2) ∓1.88017 23217 02293 (1)i	−2.13259 27935 84473 (2) ±3.76033 64837 33807 (1)i
20 160	2.56611 68976 91807 (1) ∓9.24568 62556 15094 i	2.56006 08466 86371 (1) ∓9.41207 48895 00090 i	−5.12617 77443 78178 (1) ±1.86577 61145 11518 (1)i
30 150	1.10500 69968 70304 (1) ∓5.41185 82773 81614 i	1.02118 41734 24509 (1) ∓6.86371 21678 01545 i	−2.12619 11702 94813 (1) ±1.22755 70445 18316 (1)i
40 140	6.81070 58217 30424 ∓2.77473 54068 39656 i	3.91524 75957 82218 ∓6.22540 81498 19888 i	−1.07259 53417 51264 (1) ±9.00014 35566 59544 i
50 130	5.77763 72395 07811 ∓9.92377 51947 74830 (−1)i	−2.59751 02764 67654 (−2) ∓5.86218 64953 74237 i	−5.75166 21367 43134 ±6.85456 40148 51720 i
60 120	5.89834 39694 84770	−2.94917 19847 42385 ∓5.10811 57178 32557 i	−2.94917 19847 42385 ±5.10811 57178 32557 i
70 110	6.34386 76730 64185 ±3.59666 66987 31461 (−1)i	−5.09087 38578 44618 ∓3.80223 88839 05204 i	−1.25299 38152 19567 ±3.44257 22140 32058 i
80 100	6.72908 70025 06195 ±2.88354 16209 99233 (−1)i	−6.42189 63327 31407 ∓2.03051 90227 49862 i	−3.07190 66977 47873 (−1) ±1.74216 48606 49938 i
90	6.87518 58180 20373	−6.87518 58180 20373	0

TABLE 3

θ°	tan θ	cot θ	
1	0.01745 50649 28217 58576 51289 0	57.28996 16307 59424 68727 815	89
2	0.03492 07694 91747 73050 04026 3	28.63625 32829 15603 55075 651	88
3	0.05240 77792 83041 20403 88058 2	19.08113 66877 28211 06340 675	87
4	0.06992 68119 43510 41366 69210 6	14.30066 62567 11927 91012 805	86
5	0.08748 86635 25924 00522 20186 7	11.43005 23027 61343 06721 086	85
6	0.10510 42352 65676 46251 15024	9.51436 44542 22584 92968 3971	84
7	0.12278 45609 02904 59113 42311	8.14434 64279 74594 02382 5661	83
8	0.14054 08347 02391 44683 81177	7.11536 97223 84208 74823 0566	82
9	0.15838 44403 24536 29383 88831	6.31375 15146 75043 09897 9464	81
10	0.17632 69807 08464 97347 10904	5.67128 18196 17709 53099 4418	80
11	0.19438 03091 37718 48424 31942	5.14455 40159 70310 13472 3221	79
12	0.21255 65616 70022 12525 95917	4.70463 01094 78454 23358 6235	78
13	0.23086 81911 25563 11174 81456	4.33147 58742 84155 54554 6168	77
14	0.24932 80028 43180 69162 40399	4.01078 09335 35844 71634 5715	76
15	0.26794 91924 31122 70647 25537	3.73205 08075 68877 29352 7446	75
16	0.28674 53857 58807 94004 27581	3.48741 44438 40908 65069 6224	74
17	0.30573 06814 58660 35573 45420	3.27085 26184 84140 86530 8856	73
18	0.32491 96962 32906 32615 58714	3.07768 35371 75253 40257 0291	72
19	0.34432 76132 89665 24195 72658	2.90421 08776 75822 80257 9326	71
20	0.36397 02342 66202 36135 10479	2.74747 74194 54622 27876 1664	70
21	0.38386 40350 35415 79597 14484	2.60508 90646 93801 53625 8412	69
22	0.40402 62258 35156 81132 23481	2.47508 68534 16295 82524 0013	68
23	0.42447 48162 09604 74202 35321	2.35585 23658 23752 83393 9587	67
24	0.44522 86853 08536 16392 23670	2.24603 67739 04216 05416 3321	66
25	0.46630 76581 54998 59283 00062	2.14450 69205 09558 61635 6261	65
26	0.48773 25885 65861 42277 31111	2.05030 38415 79296 21689 9011	64
27	0.50952 54494 94428 81051 37069	1.96261 05055 05150 58230 4640	63
28	0.53170 94316 61478 74807 59159	1.88072 64653 46332 01236 0838	62
29	0.55430 90514 52768 91782 07631	1.80404 77552 71423 93738 1785	61
30	0.57735 02691 89625 76450 91488	1.73205 08075 68877 29352 7446	60
31	0.60086 06190 27560 41487 86644	1.66427 94823 50517 91103 0496	59
32	0.62486 93519 09327 50978 05108	1.60033 45290 41050 35532 6733	58
33	0.64940 75931 97510 57698 20629	1.53986 49638 14582 90482 6797	57
34	0.67450 85168 42426 63214 24609	1.48256 09685 12740 25478 7157	56
35	0.70020 75382 09709 77945 85227	1.42814 80067 42114 50216 0618	55
36	0.72654 25280 05360 88589 54668	1.37638 19204 71173 53820 7210	54
37	0.75355 40501 02794 15707 39564	1.32704 48216 20410 03715 9473	53
38	0.78128 56265 06717 39706 29500	1.27994 16321 93078 78031 1030	52
39	0.80978 40331 95007 14803 69914	1.23489 71565 35051 39855 6175	51
40	0.83909 96311 77280 01176 31273	1.19175 35925 94209 95870 5308	50
41	0.86928 67378 16226 66220 00956	1.15036 84072 21009 55587 6331	49
42	0.90040 40442 97839 94512 04772	1.11061 25148 29192 87014 3482	48
43	0.93251 50861 37661 70561 21856	1.07236 87100 24682 53294 6028	47
44	0.96568 87748 07074 04595 80273	1.03553 03137 90569 50695 8833	46
45	1.00000 00000 00000 00000 00000	1.00000 00000 00000 00000 0000	45
	cot θ	tan θ	θ°

Values of the two coefficients σ_4 and σ_6 and the three functions e_r are computed for $\alpha = 10°\ (10°)\ 170°$. The results rounded off to 16D are shown in Tables 1 and 2. In these tables, (n) at the end of the number represents the factor 10^n. It should also be pointed out that whenever the sign is ambiguous, the upper one corresponds to the upper value of α and the lower with the lower value of α. The computation was carried out, with adequate guard figures, on an IBM 1620 electronic computer, using a FORTRAN II program. Checking was performed by the method described in one of the previous papers [2]. It appears that when two values of α are supplementary angles, the respective corresponding values of σ_4, σ_6, and e_r are conjugate complex numbers.

Two coefficients deserve special mention. One is σ_4 for $\alpha = 90°$, which is equal to $(4/15)\,[K(\sin 45°)]^4$; the other is σ_6 for $\alpha = 60°$ or $120°$, which is equal to $(64\sqrt{3}/315)\,[K(\sin 15°)]^6$, where $K(k)$ represents the complete elliptic integral of the first kind in Legendre form, of modulus k. The values of these coefficients to 25D are:

$$\sigma_4 = 3.15121\ 20021\ 53897\ 53821\ 76899,$$
$$\sigma_6 = 5.86303\ 16934\ 25401\ 59797\ 02134. \tag{9}$$

For the convenience of the reader, there is included in Table 3 a compilation of 25D values of the tangent and cotangent for arguments 1°(1°)89°. These data, which are required in computing the values of c, are here tabulated with the same range and precision as for the values of sine and cosine given by G. W. and R. M. Spenceley [4]. The only comparable table of decimal approximations to the tangent appears to be the relatively inaccessible 30D table of Herrmann [5].

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The Asymptotic Expansion of the Integrals Psi and Chi In Terms of Tchebycheff Polynomials

By G. T. Thompson

Hummer [3] has given the expansion of the Dawson function which is used in the calculation of line-absorbtion coefficient due to Doppler effect and damping. Psi and Chi are two integrals used to determine the Doppler broading effect on neutron cross sections in the resonance region [1]. They are functions of two variables and are given by

$$\psi(x, \theta) = \frac{1}{\sqrt{4\theta}} \int_{-\infty}^{\infty} \frac{\exp\left[-(y-x)^2/4\theta\right]}{1+y^2}\,dy$$

$$\chi(x, \theta) = \frac{1}{\sqrt{4\theta}} \int_{-\infty}^{\infty} \frac{y\exp\left[-(y-x)^2/4\theta\right]}{1+y^2}\,dy$$

and satisfy these conditions

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