Two coefficients deserve special mention. One is $\sigma_4$ for $\alpha = 90°$, which is equal to $(4/15)\,[K(\sin 45°)]^4$; the other is $\sigma_6$ for $\alpha = 60°$ or $120°$, which is equal to $(64\sqrt{3}/315)\,[K(\sin 15°)]^6$, where $K(k)$ represents the complete elliptic integral of the first kind in Legendre form, of modulus $k$. The values of these coefficients to 25D are:

$$\sigma_4 = 3.15121\ 20021\ 53897\ 53821\ 76899, \tag{9}$$
$$\sigma_6 = 5.86303\ 16934\ 25401\ 59797\ 02134.$$

For the convenience of the reader, there is included in Table 3 a compilation of 25D values of the tangent and cotangent for arguments 1°(1°)89°. These data, which are required in computing the values of $c$, are here tabulated with the same range and precision as for the values of sine and cosine given by G. W. and R. M. Spenceley [4]. The only comparable table of decimal approximations to the tangent appears to be the relatively inaccessible 30D table of Herrmann [5].

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# The Asymptotic Expansion of the Integrals Psi and Chi In Terms of Tchebycheff Polynomials

By G. T. Thompson

Hummer [3] has given the expansion of the Dawson function which is used in the calculation of line-absorbtion coefficient due to Doppler effect and damping. Psi and Chi are two integrals used to determine the Doppler broading effect on neutron cross sections in the resonance region [1]. They are functions of two variables and are given by

$$\psi(x, \theta) = \frac{1}{\sqrt{4\theta}} \int_{-\infty}^{\infty} \frac{\exp\left[-(y - x)^2/4\theta\right]}{1 + y^2}\, dy$$

$$\chi(x, \theta) = \frac{1}{\sqrt{4\theta}} \int_{-\infty}^{\infty} \frac{y \exp\left[-(y - x)^2/4\theta\right]}{1 + y^2}\, dy$$

and satisfy these conditions

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$$0 < \psi(x, \theta) \leqq 1; \qquad \psi(-x, \theta) = \psi(x, \theta)$$

and

$$-1 < \chi(x, \theta) < 1; \qquad \chi(-x, \theta) = -\chi(x, \theta).$$

Chi is related to Psi by the formula

$$\chi = 2\theta \frac{\partial \psi}{\partial x} + x\psi. \tag{1}$$

Both Psi and Chi can be expanded in a uniformly convergent series in terms of the incomplete gamma function. By making a change of the variable of integration, we have

$$\begin{aligned}\psi(x, \theta) &= \exp\left[-(x^2 - 1)/4\theta\right](4\theta)^{-1/2} \int_{1/4\theta}^{\infty} u^{-1/2} e^{-u} \exp\left[\frac{-1}{u}\left(\frac{x}{4\theta}\right)^2\right] du \\ &= (4\theta)^{-1/2} \exp\left[-(x^2 - 1)/4\theta\right] \int_{1/4\theta}^{\infty} e^{-u} u^{-1/2} \sum_{i=0}^{\infty} \frac{1}{i!}\left[\frac{1}{u}\left(\frac{x}{4\theta}\right)^2\right] du.\end{aligned}$$

Let

$$\Gamma_i(\theta) = \frac{e^{-1/4\theta}}{\sqrt{4\theta}} \int_{1/4\theta}^{\infty} u^{(-2i+1)/2} e^{-u}\, du.$$

Then

$$\Gamma_{i+1}(\theta) = \frac{2}{2i + 1}\left[(4\theta)^i - \Gamma_i(\theta)\right]$$

and

$$\psi(x, \theta) = e^{-x^2/4\theta} \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{x}{4\theta}\right) 2i\Gamma_i(\theta).$$

The incomplete gamma function is $e^{1/4\theta}$ times $\Gamma_0(\theta)$. The term by term integration is justified since the initial series converges uniformly. Similarly

$$\chi(x, \theta) = e^{-x^2/4\theta} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{x}{4\theta}\right)^{2k+1} \Gamma_{k+1}(\theta).$$

These series converge uniformly since

$$\Gamma_i(\theta) < 1.$$

However, the usual asymptotic expansion of these functions also has an interest for us here because when $\theta$ is small the convergence is slow and the accuracy is not good. For this development we write Psi as

$$\psi(x, \theta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-u^2}}{1 + (u\sqrt{(4\theta + x)})^2}\, du.$$

By substituting the expansion

$$\frac{1}{1 + (u\sqrt{(4\theta + x)})^2} = \frac{1}{2} \sum_{n=0}^{\infty} \left[\frac{(-i)^n}{(1 + ix)^{n+1}} + \frac{(i)^n}{(1 - ix)^{n+1}}\right] (u\sqrt{(4\theta)})^n \tag{2}$$

in the equation above and then integrating term by term one obtains

$$\psi(x,\theta) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-4\theta)^n}{(\sqrt{(1+x^2)})^{2n+1}} \cdot \cos\left((2n+1)\cos^{-1}(x^2+1)^{-1/2}\right) \int_0^{\infty} u^{2n} e^{-u^2}\, du$$

where the following identity was employed

$$(1+ix)^{-2n-1} + (1-ix)^{-2n-1} = 2(1+x^2)^{n+1/2} \cos\left((2n+1)\cos^{-1}(1+x^2)^{-1/2}\right).$$

Since

$$\int_0^{\infty} u^{2n} e^{-u^2}\, du = \frac{|1\cdot 3\cdot 5 \cdots (2n-1)|}{2^{n+1}} \sqrt{\pi}$$

and

$$\cos\left((2n+1)\cos^{-1}(x^2+1)^{-1/2}\right) = T_{2n+1}((1+x^2)^{-1/2})$$

where

$$T_k(x) \text{ is a Tchebycheff polynomial}$$

defined by

$$T_0(x) = 1,\ T_1(x) = x$$

and

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x),$$

we have

$$\psi(x,\theta) = \sum_{n=0}^{\infty} \frac{|1\cdot 3\cdot 5 \cdots (2n-1)|(-2\theta)^n}{(\sqrt{(x^2+1)})^{2n+1}} T_{2n+1}(1/\sqrt{(x^2+1)}).$$

Expansion (2) is permissible because of Watson's lemma [2, p. 236]. Similarly

$$\chi(x,\theta) = \sum_{n=0}^{\infty} \frac{|1\cdot 3\cdot 5 \cdots (2n-1)|(-2\theta)^n}{(\sqrt{(x^2+1)})^{2n+1}} T_{2n+1}(x/\sqrt{(x^2+1)}).$$

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