A Class of Expansions of G-functions and the Laplace Transform

By Arun Verma

1. Introduction. Several general expansions involving G-functions were given in a series of papers by Meijer [4]. Recently, Wimp and Luke [6] obtained some expansions involving G-functions which were more general than the results of Meijer. These results were obtained by generalising certain known results of Field and Wimp [3]. The object of this paper is to use the Laplace transform and its inverse to derive certain types of expansions involving G-functions. The advantage of this method, in the present context, is to show how the expansions involving G-functions follow as obvious and natural consequences of similar expansions for the generalised hypergeometric functions.*

In §3 an expansion involving G-function has been proved by using a generalisation of a result due to Niblett [5] deduced in §2. The expansion is incidentally a generalisation of some of the expansions given by Wimp and Luke,† and contains as special cases many of the expansion theorems of Meijer [4].

2. In the first instance we shall prove the following generalisation of a result due to Niblett [5].

$$
\begin{aligned}
{}_{p+s}F_q\left[\begin{matrix}(a_p),\,(b_s);\,xw\\(c_q)\end{matrix}\right] &= h\sum_{n=0}^{\infty}\frac{[h-n\alpha+1]_{n-1}[(b_s)]_n[(e_u)]_n(-x)^n}{n![(c_q)]_n}\\
(1)\qquad &\times {}_{p+2}F_{u+2}\left[\begin{matrix}(a_p),\,1+h(1-\alpha)^{-1},\,-n;\,w\\(e_u),\,h(1-\alpha)^{-1},\,h-n\alpha+1\end{matrix}\right]\\
&\times {}_{s+u+1}F_q\left[\begin{matrix}(b_s)+n,\,(e_u)+n,\,h+n(1-\alpha);\,x\\(c_q)+n\end{matrix}\right],
\end{aligned}
$$

provided $p + s \leqq q$, or $p + s = 1 + q$, and $|xw| < 1$, $s + u + 1 \leqq q$, or $s + u = q$ and $|x| < 1$, and the series of the hypergeometric functions on the right hand is absolutely convergent. To prove (2.1) we use a simple extension of the method used earlier by Chaundy [1] for proving a similar result.

Comparing the coefficient of $[(a_p)]_N w^N/N!$ on both the sides of (2.1), we get

$$
\begin{aligned}
\frac{[(b_s)]_N x^N}{[(c_q)]_N} &= \{h+N(1-\alpha)\}\sum_{n=N}^{\infty}\frac{[h-n\alpha+1]_{n-1}[(b_s)]_n[(e_u)]_n(-x)^n}{n![(c_q)]_n}\\
&\times\frac{[-n]_N}{[h-n\alpha+1]_N}\,{}_{s+u+1}F_q\left[\begin{matrix}(b_s)+n,\,(e_u)+n,\,h+n(1-\alpha);\,x\\(c_q)+n\end{matrix}\right].
\end{aligned}
$$

Writing $n = N + r$, we find that this reduces to

$$
\begin{aligned}
1 &= \{h+N(1-\alpha)\}\sum_{r=0}^{\infty}\frac{[h+N(1-\alpha)+1-\alpha]_{r-1}[(b_s)+N]_r[(e_u)+N]_r(-x)^r}{r![(c_q)+N]_r}\\
&\times {}_{s+u+1}F_q\left[\begin{matrix}(b_s)+N+r,\,(e_u)+N+r,\,h+(1-\alpha)(N+r);\,x\\(c_q)+N+r\end{matrix}\right].
\end{aligned}
$$

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* For the notation and the properties of G-functions see Erdélyi [2].

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The term independent of x on the right hand side is seen to be unity. It thus remains to show that the coefficient of any positive power of x vanishes on the right, i.e., when $M > 0$,

$$\frac{[(b_s)+N]_M[(e_u)+N]_M}{[(c_q)+N]_M}\sum_{r=0}^{M}\frac{(-)^r[h+N(1-\alpha)+1-r\alpha]_{M-1}}{r![M-r]!}=0.$$

This, however, is the coefficient of x^{M-1} in

$$\frac{[(b_s)+N]_M[(e_u)+N]_M}{M[(c_q)+N]_M}(1-x)^{-h-N(1-\alpha)-1}[1-(1-x)^{\alpha}]^M,$$

in which the lowest term is x^M. This completes the formal proof of (2.1). The rearrangement of the infinite series is justified due to absolute convergence.

This result reduces to one proved by Niblett [5] for $w = 1$ and $u = q$. And for $\alpha = 0$, this gives us a result due to Meijer [4] on assuming $u = q$, $(e_u) = (c_q)$ and after suitable adjustments of the parameters.

3. Next, we generalise (2.1) for the G-functions in the following form:

$$G^{h+s+t,1+v}_{1+l+q_1+W,h+s+m}\left(xw\,\middle|\,\begin{matrix}1,(k_l),(c_q),(\delta_W)\\(a_p),(b_s),(g_m)\end{matrix}\right)$$

(1)
$$= h\Gamma[(a_p);(e_u),(\delta_W)]\sum_{n=0}^{\infty}\frac{1}{n!\Gamma[1-\alpha n+h]}$$
$$\times\ {}_{p+2}F_{W+u+2}\left[\begin{matrix}(a_p),1+h(1-\alpha)^{-1},-n;1/w\\(\delta_W),(e_u),h(1-\alpha)^{-1},h-n\alpha+1\end{matrix}\right]$$
$$\times\ G^{1+s+u+t,1+v}_{1+l+q,1+s+u+m}\left(x\,\middle|\,\begin{matrix}1-n,(k_l),(c_q)\\(b_s),(e_u),h-\alpha n,(g_m)\end{matrix}\right),$$

provided

$v \leqq l$, $t \leqq m$, $l + m + q + W < 1 + h + s + 2t + 2v$, $|\arg xw| <$
$\frac{1}{2}\pi(1 + h + s + 2t + 2v - l - m - q - W)$, $t + v < 2 + 2l +$
$2m + 2q + s + u$, $|\arg x| < \frac{1}{2}\pi(2 + 2l + 2m + 2q + s + u - t - v)$,

and the series on the right hand side has a meaning.

We prove (3.1) by mathematical induction. We suppose that (3.1) is true for some fixed values of l, m, p, q, s, t, u, v and W. To effect the induction with respect to v multiply both sides by z^{-f_v+1}, replace x by xz and take the Laplace transform with respect to z on both the sides. Then using the known result

(2)
$$\int_0^{\infty}e^{-y}y^{-\alpha}G^{m,n}_{p,q}\left(xy\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right)dy = G^{m,n+1}_{p+1,q}\left(x\,\middle|\,\begin{matrix}\alpha,(a_p)\\(b_q)\end{matrix}\right),$$

provided
(i) $p + q < 2(m + n)$, $|\arg x| < \pi(m + n - \frac{1}{2}p - \frac{1}{2}q)$,
(ii) $Rl\alpha < Rlb_h + 1$, $h = 1, 2, \cdots, m$;

on both the sides, we get a relation in which v has been replaced by $v + 1$.

Further, to effect the induction with respect to m multiply both sides by $z^{g_{m+1}-1}$, replace x by x/z and take the inverse Laplace transform with respect to z on both the sides. Then using the known result

$$(3) \qquad \frac{1}{2\pi i}\int_c e^t t^{\alpha-1} G_{p,q}^{m,n}\left(\frac{x}{t}\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right) dt = G_{p,q+1}^{m,n}\left(x\,\middle|\,\begin{matrix}(a_p)\\(b_q),\,\alpha\end{matrix}\right),$$

provided $Rl\alpha > 0$, and the conditions (3.2-i) and (3.2-ii) hold; on both the sides, we get a relation in which m has been replaced by $m + 1$.

Similarly, the induction with respect to l, N, p, q, s, t, u and W can be effected. Since, for $l = m = t = v = W = 0$, (3.1) reduces to the relation (2.1), the result is established completely.*

Next, using first the relation [2; 5.3.1 (9)]

$$G_{p,q}^{m,n}\left(x\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right) = G_{q,p}^{n,m}\left(x^{-1}\,\middle|\,\begin{matrix}1-(b_q)\\1-(a_p)\end{matrix}\right),$$

and then

$$G_{p+1,q+1}^{m+1,k}\left(x\,\middle|\,\begin{matrix}(a_p),\,1-n\\1,\,(b_q)\end{matrix}\right) = (-)^n G_{p+1,q+1}^{m,k+1}\left(x\,\middle|\,\begin{matrix}1-n,\,(a_p)\\(b_q),\,1\end{matrix}\right),$$

which is apparent from the definition of the G-function, on both the sides of (3.1) one can get an expansion of $G(\lambda x)$ in a series of the products of $G(x)$ and $F(\lambda)$.

It is worth noting that by taking $\alpha = 0$, $g_m = 1$, $a_p = h$, $c_q = h$, we get a result which is essentially the sixth theorem of Wimp and Luke [6], which in its turn contains as a particular case Theorem 6 of Meijer [4].

One can easily extend a result due to Carlitz and Alsalam [7] to G-functions by using the above technique.

Added in proof. In (3.1) taking $a_1 = t$ and then letting $t \to 0$, we get the sum of an infinite series of G-functions in terms of products of gamma functions, a result which also generalises the known formula due to MacRobert and Ragab [*Math. Z.*, v. 78, 1962] and is perhaps a solitary example of its type for the G-functions.

Department of Mathematics,
The University, Gorakhpur, India.

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* The conditions on the parameters arise due to the particular method followed and can be waived off by analytic continuation.