135[X].—RUFUS ISAACS, *Differential Games*, John Wiley & Sons, Inc., New York, 1965, xxii + 384 pp., 24 cm. Price $15.00.

Although the theory and application of differential games (and control theory) has received much attention since 1950, I feel it will be worthwhile to begin with a short description of the types of problems dealt with in these fields, and in particular in this book.

In a quite general case, the evolution of a system, whose state we assume is determined by the state variables $x = (x^1, \cdots, x^n)$, is governed by a differential equation (1) $\dot{x} = F(x, \phi, \psi)$ with the initial condition $x(0) = x_0$, where $\phi = (\phi^1, \cdots, \phi^s)$ and $\psi = (\psi^1, \cdots, \psi^r)$ are termed controls. There will be two players, one controlling the function ϕ, which may have values in a given set in Euclidean s-dimensional space (E^s); the other controlling ψ within a given set in E^r. There is also given a manifold C, the terminating manifold, such that the game ends when a trajectory of (1) enters C, and a payoff functional defined on the space of trajectories of (1). The game proceeds with one player controlling ϕ in such a way as to generate a trajectory which maximizes payoff, the other player utilizing ψ to try to minimize payoff.

At this point one should notice that if at the offset each player knows that the other will play optimally, the functions ϕ and ψ can be computed as functions of *time* alone, and the value of the game becomes merely a function of the starting position x_0. However, if a player may err, this is no longer the case. In order that the opposing player may use the error to his benefit, he must be able to perceive it, i.e., to make measurements on the state of the system as the game progresses. With the exception of Chapter 12, the author assumes that at each instant of time during the course of play, both players have complete knowledge of the present state of the system; he defines a *strategy* as a determination of the controls ϕ and ψ as functions of the state. Since it is assumed that the players do not err, but play optimally, the determination of a strategy implies that the game has been solved for arbitrary initial positioning x_0.

I feel that these important distinctions in possible types of games have been passed over somewhat lightly. For instance, the casual reader may not notice that a proposed method of play in a game of two players (example 8.1.1, p. 202) requires one player to have "memory", i.e., the value of the control depends on past history of the state.

As the author remarks, the theory of differential games grew from solving problems, and this is the approach taken in the book. Little time or effort is spent on theorem proving, instead many diversified types of problems are formulated, often completely solved, and a theory introduced which stems from the method of solution.

The book begins at a leisurely pace, with the first three chapters being accessible to a person with little mathematical background. Chapters 1 and 2 are of an introductory nature while Chapter 3 deals with discrete games. The variety of fascinating problems formulated, and often solved, in these chapters alone should delight a wide audience of readers.

In Chapter 2, pages 41–43, the need for having the vectogram (the set of values $\{F(x, \phi, \psi)\}$ as ϕ, ψ take on all admissible values) convex for each x in order that a solution exist, is illustrated by example. Here the author has led the reader in a very natural way to the basis of the deep existence theorems of control theory.

The mathematical theory begins in Chapter 4. The approach taken is to show that the payoff, or value $V(x)$, of a game starting at the arbitrary initial state x, satisfies what has become known as the "Bellman equation" which somewhat resembles a partial differential equation of Hamilton-Jacobi type. The solutions of such equations often exhibit extreme changes in neighborhoods of certain surfaces in state space called singular surfaces by the author, or switching surfaces in control theory. The majority of the remainder of the book is concerned with showing, mainly by example, the types of behavior which solutions may exhibit near these surfaces, in classifying the surfaces and in solving problems using the concepts introduced.

On the whole, the printing of the book is very good. There are only occasional minor errors, e.g., the rightmost vectogram for the player P in Figure 3.3.1, page 51, is in error and will not yield the shown solution.

The wide range of possible applications of differential games is exemplified in the many examples discussed and solved throughout the text. While obtaining solutions to these intriguing problems, the author has done an excellent job in providing insight into the deep mathematical theories which exist and the difficulties which must still be overcome.

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136[X].—Ben Noble, *Numerical Methods*, Oliver & Boyd Ltd., Edinburgh, Scotland, 1964. Volume 1, *Iteration, Programming and Algebraic Equations*, xii + 156 pp., 19 cm. Price $2.75. Volume 2, *Differences, Integration and Differential Equations*, viii + 372 pp., 19 cm. Price $3.00.

A reasonable knowledge of numerical analysis should be possessed by every engineer, scientist or applied mathematician. A great many books have been recently published in an attempt to fill the demand for this knowledge, particularly at an elementary level. Many of these books combine numerical analysis and computer programming, using a problem-oriented language like FORTRAN. These books are frequently disappointing, particularly if you have read the publisher's claims on the dust jacket before you read the author's preface.

It is a pleasure to report that these two volumes under review have accomplished their stated purpose and constitute an excellent elementary introduction to the most commonly used numerical methods. The first chapter of Vol. I sets the general level of the work by presenting a clear concise account of several topics such as round off, absolute and relative errors, error analysis and control, etc. The treatment is both practical and elementary. This is followed with chapters on iteration methods, elementary programming, linear equations, and matrix methods. The emphasis is placed on presenting a few methods in some detail. The chapter on programming uses a problem-oriented language, but does not try to teach FORTRAN or ALGOL. The language is used to illustrate how a source language is used without going into the vast amount of detail necessary to present an existing source language.