Correction of Stability Curves in Hill-Meissner's Equation

By Chikara Sato

1. Introduction. It is well known that the Hill-Meissner equation has peculiar transition curves from stability to instability [1]. This differential equation has the form

$$\ddot{x} + (\omega^2 + \alpha^2 \operatorname{sgn} \cos t)x = 0, \tag{1}$$

where $\ddot{x} = d^2x/dt^2$, ω^2 and α^2 are real constants, and $\operatorname{sgn} z = 1$ for $z \geqq 0$, $\operatorname{sgn} z = -1$ for $z < 0$. Equation (1) is a special case of Hill's equation, which is a differential equation with a coefficient of period 2π [2]. The transition curves of equation (1) were given by Meissner for the case $\omega^2 > \alpha^2 > 0$ in 1918 [1], and later by van der Pol and Strutt for the general case, in 1928 [3]. Since then, these transition curves [3] have been referred to in many papers, without correction (e.g. [4], [5]). The author has noticed that some correction is necessary for those transition curves, since there exists appreciable error in at least several points on the published curves. From these reasons, we present more accurate transition curves obtained by using a digital computer.

It is easy to write solutions $x(t)$ and $\dot{x}(t)$ from $t = 0$ to $t = \pi$, and from $t = \pi$ to $t = 2\pi$ respectively. Combining these solutions at $t = \pi$ so that $x(t)$ and $\dot{x}(t)$ are continuous, we have solutions $x(2\pi)$ and $\dot{x}(2\pi)$ as a linear combination of the initial conditions $x(0)$ and $\dot{x}(0)$. In short, $x(0)$ and $\dot{x}(0)$ are transformed into $x(2\pi)$ and $\dot{x}(2\pi)$ by a linear transformation. The characteristic equation of this transformation is given by the following equation [3]:

$$\lambda^2 - 2f(\omega^2, \alpha^2)\lambda + 1 = 0,$$

where

$$f(\omega^2, \alpha^2) = \cos\gamma_1 \cos\gamma_2 - \frac{1}{2}\left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1}\right) \sin\gamma_1 \sin\gamma_2, \qquad \text{for } \omega^2 > \alpha^2 > 0,$$

$$f(\omega^2, \alpha^2) = \cos\gamma_1 \cosh\gamma_3 - \frac{1}{2}\left(\frac{\gamma_1}{\gamma_3} - \frac{\gamma_3}{\gamma_1}\right) \sin\gamma_1 \sinh\gamma_3, \qquad \text{for } \omega^2 < \alpha^2,$$

and

$$\gamma_1^2 = \pi^2(\omega^2 + \alpha^2), \qquad \gamma_2^2 = \pi^2(\omega^2 - \alpha^2), \qquad \gamma_3^2 = \pi^2(\alpha^2 - \omega^2).$$

If the characteristic root λ satisfies $|\lambda| \leqq 1$ then the solutions are stable and if $|\lambda| > 1$ the solutions are unstable. Since the function $f(\omega^2, \alpha^2)$ is real and continuous for all real ω^2 and α^2 including near $\omega^2 = \alpha^2$, the critical stability condition $|\lambda| = 1$ is equivalent to the following equation

Received April 20, 1965.

$$|f(\omega^2, \alpha^2)| = 1. \tag{2}$$

The transition curves from stability to instability are obtained by solving equation (2). A digital computer K-1 which was constructed at Keio University in 1959 (with a memory of 2200 words) was used for solving equation (2).

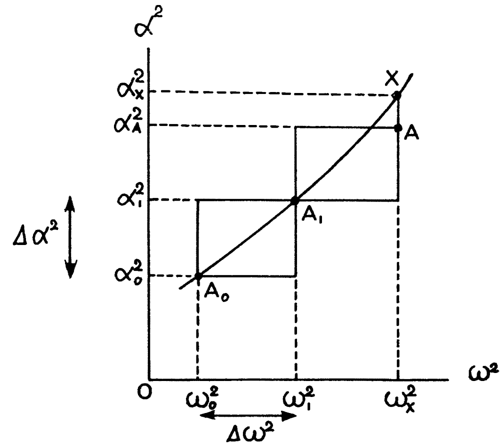

FIGURE 1. Searching method for a transition curve in the (ω^2, α^2) plane. Using two exact points A_0 and A_1, the point X is obtained by iterations.

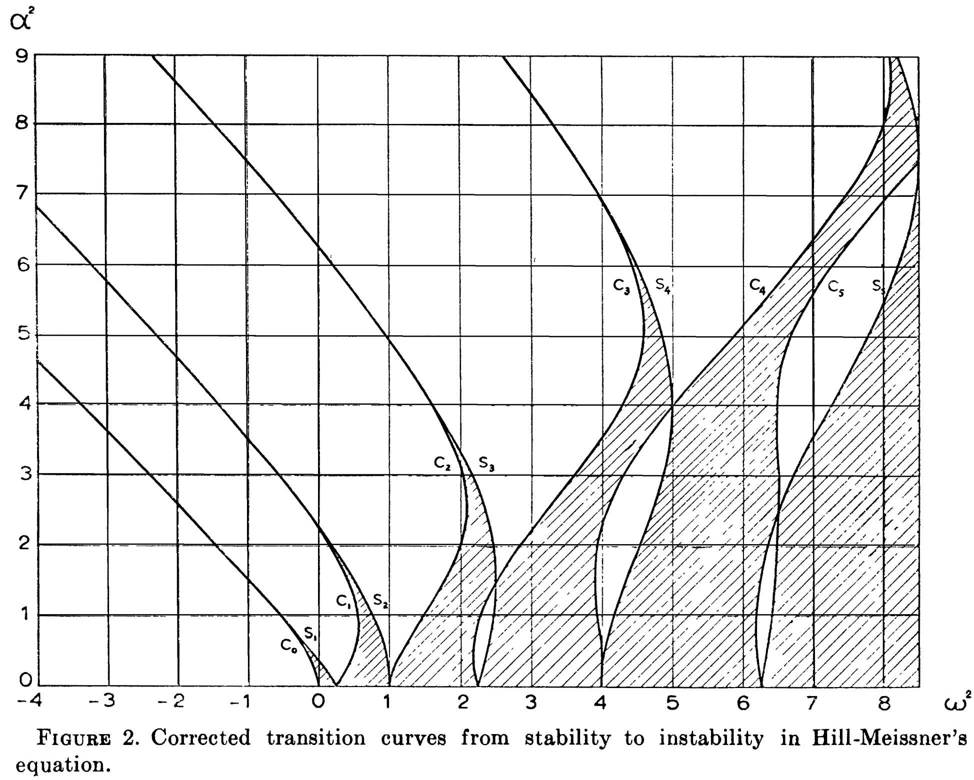

FIGURE 2. Corrected transition curves from stability to instability in Hill-Meissner's equation.

TABLE

C_0		S_1	
α^2	ω^2	α^2	ω^2
+0.00000	+0.00000	+0.00000	+0.25000
+0.30000	−0.06949	+0.20000	+0.11483
+0.50000	−0.17652	+0.40000	−0.03333
+0.70000	−0.31234	+0.60000	−0.19118
+0.90000	−0.46565	+0.80000	−0.35632
+1.10000	−0.62991	+1.00000	−0.52703
+1.30000	−0.80139	+1.20000	−0.70205
+1.50000	−0.97787	+1.40000	−0.88047
+1.70000	−1.15798	+1.60000	−1.06160
+1.90000	−1.34082	+1.83065	−1.27321
+2.10000	−1.52577	+2.06130	−1.48717
+2.30000	−1.71241	+2.29195	−1.70302
+2.50000	−1.90043	+2.52261	−1.92045
+2.70000	−2.08958	+2.75326	−2.13919
+2.90000	−2.27969	+2.98391	−2.35903
+3.10000	−2.47061	+3.21456	−2.57981
+3.30000	−2.66224	+3.44522	−2.80140
+3.50000	−2.85448	+3.67587	−3.02370
+3.70000	−3.04725	+3.90652	−3.24661
+3.90000	−3.24050	+4.13718	−3.47006
+4.10000	−3.43417	+4.36783	−3.69399
+4.30000	−3.62822	+4.59848	−3.91835
+4.50000	−3.82260	+4.82913	−4.14309
+4.70000	−4.01730	+5.05979	−4.36817
+4.90000	−4.21227	+5.29044	−4.59357

C_1		S_2	
α^2	ω^2	α^2	ω^2
+0.00000	+0.25000	+0.00000	+1.00000
+0.10000	+0.31139	+0.14000	+0.99510
+0.20000	+0.36781	+0.24000	+0.98565
+0.30000	+0.41855	+0.34000	+0.97133
+0.40000	+0.46285	+0.44000	+0.95226
+0.50000	+0.50000	+0.54000	+0.92857
+0.70000	+0.55027	+0.64000	+0.90046
+0.90000	+0.56597	+0.74000	+0.86812
+1.10000	+0.54716	+1.10000	+0.72008
+1.30000	+0.49728	+1.30000	+0.61921
+1.50000	+0.42152	+1.50000	+0.50721
+1.70000	+0.32521	+1.70000	+0.38563
+1.90000	+0.21291	+1.90000	+0.25581
+2.10000	+0.08818	+2.10000	+0.11892
+2.30000	−0.04630	+2.30000	−0.02405
+2.50000	−0.18853	+2.50000	−0.17227
+2.70000	−0.33704	+2.70000	−0.32504
+2.90000	−0.49070	+2.90000	−0.48176
+3.10000	−0.64865	+3.10000	−0.64193
+3.30000	−0.81022	+3.30000	−0.80513
+3.50000	−0.97489	+3.50000	−0.97100

C_1		S_2	
α^2	ω^2	α^2	ω^2
+3.70000	−1.14224	+3.70000	−1.13925
+3.90000	−1.31193	+3.90000	−1.30961
+4.10000	−1.48366	+4.10000	−1.48186
+4.30000	−1.65722	+4.30000	−1.65580
+4.50000	−1.83239	+4.50000	−1.83127
+4.70000	−2.00902	+4.70000	−2.00813
+4.90000	−2.18695	+4.90000	−2.18624
+5.10000	−2.36607	+5.10000	−2.36551
+5.30000	−2.54627	+5.30000	−2.54582
+5.50000	−2.72745	+5.50000	−2.72709
+5.70000	−2.90954	+5.70000	−2.90924
+5.90000	−3.09246	+5.90000	−3.09221
+6.10000	−3.27614	+6.10000	−3.27594
+6.30000	−3.46052	+6.30000	−3.46036
+6.50000	−3.64557	+6.50000	−3.64543
+6.70000	−3.83122	+6.70000	−3.83111
+6.90000	−4.01745	+6.90000	−4.01735
+7.10000	−4.20420	+7.10000	−4.20412

C_2		S_3	
α^2	ω^2	α^2	ω^2
+0.00000	+1.00000	+0.00000	+2.25000
+0.10000	+0.99749	+0.10000	+2.27206
+0.30000	+1.06291	+0.30000	+2.31923
+0.50000	+1.15794	+0.50000	+2.36659
+0.70000	+1.27502	+0.70000	+2.41035
+0.90000	+1.40191	+0.90000	+2.44758
+1.10000	+1.53070	+1.10000	+2.47598
+1.30000	+1.65579	+1.30000	+2.49385
+1.50000	+1.77255	+1.50000	+2.50000
+1.70000	+1.87666	+1.70000	+2.49368
+2.00000	+2.00000	+1.90000	+2.47457
+2.20000	+2.05498	+2.10000	+2.44272
+2.40000	+2.08530	+2.60000	+2.31013
+2.60000	+2.09035	+2.80000	+2.23768
+2.80000	+2.07119	+3.00000	+2.15536
+3.00000	+2.03014	+3.20000	+2.06401
+3.20000	+1.97010	+3.40000	+1.96445
+3.40000	+1.89400	+3.60000	+1.85748
+3.60000	+1.80448	+3.80000	+1.74380
+3.80000	+1.70377	+4.00000	+1.62409
+4.00000	+1.59371	+4.20000	+1.49894
+4.20000	+1.47575	+4.40000	+1.36888
+4.40000	+1.35109	+4.60000	+1.23439
+4.60000	+1.22066	+4.80000	+1.09589
+4.80000	+1.08523	+5.00000	+0.95376
+5.00000	+0.94544	+5.20000	+0.80832
+5.20000	+0.80179	+5.40000	+0.65987
+5.40000	+0.65473	+5.60000	+0.50868

C_2		S_3	
α^2	ω^2	α^2	ω^2
+5.60000	+0.50461	+5.80000	+0.35497
+5.80000	+0.35173	+6.00000	+0.19896
+6.00000	+0.19637	+6.20000	+0.04083
+6.20000	+0.03875	+6.40000	−0.11925
+6.40000	−0.12093	+6.60000	−0.28115
+6.60000	−0.28250	+6.80000	−0.44472
+6.80000	−0.44582	+7.00000	−0.60984
+7.00000	−0.61074	+7.20000	−0.77640
+7.20000	−0.77714	+7.40000	−0.94431
+7.40000	−0.94492	+7.60000	−1.11348
+7.60000	−1.11398	+7.80000	−1.28382
+7.80000	−1.28423	+8.00000	−1.45526
+8.00000	−1.45560	+8.20000	−1.62774
+8.20000	−1.62802	+8.40000	−1.80118
+8.40000	−1.80141	+8.60000	−1.97554
+8.60000	−1.97573	+8.80000	−2.15076
+8.80000	−2.15092	+9.00000	−2.32679
+9.00000	−2.32693	+9.20000	−2.50359
+9.20000	−2.50371		

C_3		S_4	
α^2	ω^2	α^2	ω^2
+0.00000	+2.25000	+0.00000	+4.00000
+0.20000	+2.21242	+0.60000	+4.04002
+0.40000	+2.18893	+0.80000	+4.10807
+0.60000	+2.18614	+1.00000	+4.16838
+0.80000	+2.20945	+1.20000	+4.23342
+1.00000	+2.26122	+1.40000	+4.30279
+1.20000	+2.34012	+1.60000	+4.37456
+1.40000	+2.44202	+1.80000	+4.44645
+1.50000	+2.50000	+2.00000	+4.52745
+1.60000	+2.56182	+2.20000	+4.60409
+1.80000	+2.69474	+2.40000	+4.67567
+2.00000	+2.83685	+2.60000	+4.74637
+2.20000	+2.98508	+2.80000	+4.81064
+2.40000	+3.13701	+3.00000	+4.86644
+2.60000	+3.29068	+3.20000	+4.91293
+2.80000	+3.44443	+3.40000	+4.94900
+3.00000	+3.59668	+3.60000	+4.97250
+3.20000	+3.74587	+4.00000	+5.00000
+3.40000	+3.89030	+4.30000	+4.99291
+3.60000	+4.02808	+4.50000	+4.96663
+3.80000	+4.15699	+4.60000	+4.95036
+4.00000	+4.27451	+4.70000	+4.93267
+4.20000	+4.37784	+4.80000	+4.91242
+4.50000	+4.50000	+5.10000	+4.83689
+4.70000	+4.55598	+5.30000	+4.77495
+4.90000	+4.58986	+5.50000	+4.70444
+5.10000	+4.60151	+5.70000	+4.62599
+5.30000	+4.59192	+5.90000	+4.54019

TABLE—*Continued*

C_3		S_4	
α^2	ω^2	α^2	ω^2
+5.50000	+4.56296	+6.10000	+4.44761
+5.70000	+4.51693	+6.30000	+4.34882
+5.90000	+4.45613	+6.50000	+4.24431
+6.10000	+4.38273	+6.70000	+4.13456
+6.30000	+4.29860	+6.90000	+4.01999
+6.50000	+4.20530	+7.10000	+3.90100
+6.60000	+4.15563	+7.30000	+3.77793
+6.70000	+4.10413	+7.50000	+3.65112
+6.80000	+4.05094	+7.70000	+3.52085
+6.90000	+3.99616	+7.90000	+3.38738
+7.00000	+3.93990	+8.10000	+3.25095
+7.10000	+3.88225	+8.30000	+3.11178
+7.20000	+3.82330	+8.50000	+2.97006
+7.30000	+3.76312	+8.70000	+2.82597
+7.40000	+3.70179	+8.90000	+2.67966
+7.50000	+3.63936	+9.10000	+2.53129
+7.60000	+3.57591		
+7.70000	+3.51148		
+7.90000	+3.37988		
+8.10000	+3.24493		
+8.30000	+3.10692		
+8.50000	+2.96613		
+8.70000	+2.82277		
+8.90000	+2.67706		
+9.10000	+2.52917		

C_4		S_5	
α^2	ω^2	α^2	ω^2
+0.00000	+4.00000	+0.00000	+6.25000
+0.10000	+4.00206	+0.16000	+6.23069
+0.30000	+3.99444	+0.36000	+6.21031
+0.50000	+3.98500	+0.56000	+6.19522
+0.70000	+3.97180	+0.76000	+6.18668
+0.90000	+3.95608	+0.96000	+6.18584
+1.10000	+3.93958	+1.16000	+6.19369
+1.30000	+3.92455	+1.36000	+6.21103
+1.50000	+3.91386	+1.56000	+6.23838
+1.70000	+3.91094	+1.76000	+6.27596
+1.90000	+3.91959	+1.96000	+6.32372
+2.10000	+3.94355	+2.10000	+6.36303
+2.30000	+3.98583	+2.20000	+6.39395
+2.50000	+4.04810	+2.30000	+6.42715
+2.70000	+4.13027	+2.40000	+6.46253
+2.90000	+4.23082	+2.50000	+6.50000
+3.10000	+4.34727	+2.60000	+6.53941
+3.30000	+4.47689	+2.70000	+6.58069
+3.50000	+4.61708	+2.80000	+6.62370
+3.70000	+4.76558	+2.90000	+6.66831
+3.90000	+4.92056	+3.00000	+6.71439

TABLE—*Continued*

C_4		S_5	
α^2	ω^2	α^2	ω^2
+4.00000	+5.00000	+3.10000	+6.76184
+4.10000	+5.08051	+3.17300	+6.78699
+4.30000	+5.24424	+3.37300	+6.89301
+4.50000	+5.41076	+3.57300	+6.99893
+4.70000	+5.57930	+3.77300	+7.10591
+4.90000	+5.74915	+3.97300	+7.21377
+5.10000	+5.91973	+4.17300	+7.32182
+5.30000	+6.09047	+4.37300	+7.42915
+5.50000	+6.26084	+4.57300	+7.53476
+5.70000	+6.43029	+4.77300	+7.64090
+5.90000	+6.59823	+4.97300	+7.74629
+6.10000	+6.76400	+5.17300	+7.84742
+6.30000	+6.92684	+5.37300	+7.94362
+6.50000	+7.08583	+5.57300	+8.03414
+6.70000	+7.23987	+5.77300	+8.11812
+6.90000	+7.38763	+5.97300	+8.19567
+7.10000	+7.52745	+6.17300	+8.26833
+7.30000	+7.65738	+6.37300	+8.33172
+7.50000	+7.77514	+6.57300	+8.38522
+7.70000	+7.87823	+6.80000	+8.43481
+8.00000	+8.00000	+7.00000	+8.46666
+8.20000	+8.05634	+7.20000	+8.48799
+8.40000	+8.09156	+7.40000	+8.49866
+8.60000	+8.10577	+7.50000	+8.50000
+8.80000	+8.10007	+7.60000	+8.49867
+9.00000	+8.07624	+7.80000	+8.48815
+9.20000	+8.03639	+8.00000	+8.46738
+9.40000	+7.98263	+8.60000	+8.34765
+9.60000	+7.91694	+8.80000	+8.29028
+9.80000	+7.84101	+9.00000	+8.22508
		+9.20000	+8.15261
		+9.40000	+8.07339

C_5

α^2	ω^2
+0.00000	+6.25000
+0.07000	+6.25907
+0.27000	+6.28664
+0.47000	+6.31566
+0.67000	+6.34500
+0.87000	+6.37369
+1.07000	+6.40084
+1.27000	+6.42567
+1.47000	+6.44755
+1.67000	+6.46597
+1.87000	+6.48055
+2.07000	+6.49107
+2.10000	+6.49231
+2.20000	+6.49573
+2.30000	+6.49812

TABLE—*Continued*

C_5

α^2	ω^2
+2.40000	+6.49953
+2.50000	+6.50000
+2.60000	+6.49955
+2.70000	+6.49828
+2.80000	+6.49624
+2.90000	+6.49355
+3.00000	+6.49030
+3.27500	+6.48428
+3.47500	+6.47395
+3.67500	+6.46800
+3.87500	+6.46779
+4.07500	+6.47600
+4.27500	+6.49435
+4.47500	+6.52014
+4.67500	+6.56315
+4.87500	+6.62436
+5.07500	+6.70388
+5.27500	+6.80085
+5.47500	+6.91200
+5.67500	+7.03541
+5.87500	+7.17084
+6.07500	+7.31600
+6.27500	+7.46901
+6.47500	+7.62840
+6.67500	+7.79311
+6.87500	+7.96261
+7.07000	+8.12488
+7.27000	+8.29819
+7.47000	+8.47354
+7.50000	+8.50000
+7.67000	+8.65052

2. Method of Computation. The method of obtaining the transition curves will now be described. Let A_0 and A_1 be two exact points on a transition curve in the (ω^2, α^2) plane, as shown in Fig. 1. It is supposed that these points are known beforehand by hand computation. Starting from these points A_0 and A_1, a third point is predicted by linear extrapolation. The first prediction is the point A which lies on a straight line through the points A_0 and A_1, with $\overline{A_0A_1} = \overline{A_1A}$. A more exact point X is then located. This lies on a vertical line through the point A if $\Delta\omega^2 > \Delta\alpha^2$. On the other hand if $\Delta\omega^2 \leqq \Delta\alpha^2$, the point X lies on a horizontal line through the point A. To converge toward an exact point, Newton's approximation method is used.

By the computer program, consisting of 281 words, it required 3.2 hours to obtain the 468 points listed in the Table. Comparing the old curves given by van der Pol with the new curves of Fig. 2, an appreciable difference is seen in Fig. 3.

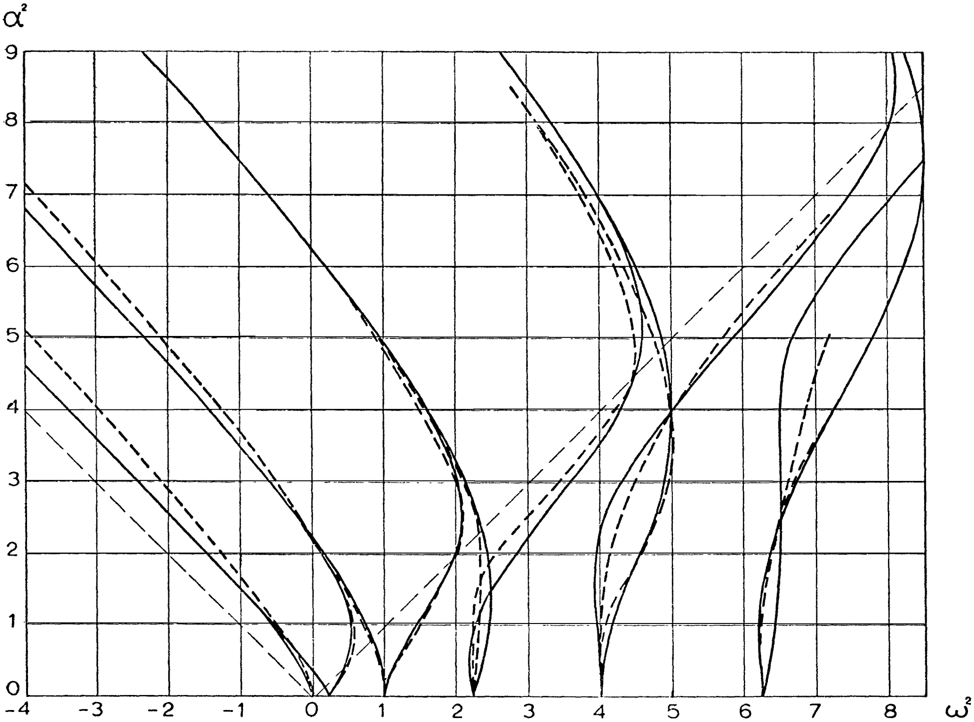

FIGURE. 3. Comparison between transition curves by van der Pol (1928) and corrected transition curves (in Fig. 2). -----: curves by van der Pol, ——: corrected curves.

3. Acknowledgment. The author wishes to thank Professors J. Yamauti and S. Ura, School of Engineering, Keio University, Tokyo, for their helpful suggestions to this work. He also thanks Mr. Y. Yushita and Mr. K. Saito, who programmed the main part of the computation.

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1. E. MEISSNER, "Ueber Schüttelerscheinungen in Systemen mit periodisch veränderlicher Elastizität," *Schweiz. Bauzeit.*, v. 72, 14, 1918, pp. 95–98.
2. WHITTAKER & WATSON, *A Course of Modern Analysis*, Cambridge, 1935, pp. 404–428.
3. B. VAN DER POL & M. J. O. STRUTT, "On the stability of the solutions of Mathieu's equation," *Philos. Mag.*, v. 5, 1928, pp. 18–38.
4. N. MINORSKY, *Introduction to Non-Linear Mechanics. Topological Methods, Analytic Methods, Non-Linear Resonance. Relaxation Oscillations*, J. W. Edwards, Ann Arbor, Michigan, 1947, pp. 359–369. MR **8**, 583.
5. L. BRILLOUIN, *Wave Propagation in Periodic Structures*, 2nd ed. Dover, New York, 1953, pp. 180–186.