On Divisors of Odd Perfect Numbers

By Joseph B. Muskat

A perfect number is a positive integer the sum of whose divisors is equal to twice the number itself. Twenty-three even perfect numbers have been discovered to date [2]. No odd perfect number has yet been found, but various restrictions which an odd perfect number must satisfy have been established. For a summary, see [7].

For a perfect number n, $\sigma(n) = 2n$, where $\sigma(n)$ denotes the sum of the divisors of n. Let

$$n = \prod f_j^{e_j},$$

where the f_j are distinct primes. Since σ is a multiplicative function [8, p. 88],

$$2n = 2\prod f_j^{e_j} = \sigma(n) = \prod \sigma(f_j^{e_j}). \tag{1}$$

(2) *Any divisor of the right side of* (1) *must divide* $2n$

is an immediate consequence of (1). For example if 9, but not 27, divides n, then $\sigma(3^2) = 13$ divides n.

Euler deduced from (1) that n must be of the form

$$n = p^a \prod_{i=1}^{r} q_i^{2b_i}, \qquad \text{where} \qquad p \equiv a \equiv 1 \pmod 4 \tag{3}$$

and p and the q_i denote distinct primes [1, pp. 14–15]. Kühnel [5] and others have proved that $r \geqq 5$.

Using these and other results, Kanold showed that there are no odd perfect numbers less than 10^{20} [3]. This superseded a bound of 10^{18}, obtained by the author [8, p. 359b] with the help of the following:

(4) *Any odd perfect number must be divisible by a prime power greater than* 10^8.

Ore studied numbers whose harmonic means are integers, and showed that perfect numbers have this property [9]. W. H. Mills demonstrated that any odd number with an integral harmonic mean must have a prime power factor greater than 10^7. This bound in Mills' (unpublished) calculation arose from the limited range of D. N. Lehmer's factor table [6] which Mills utilized. The author (as a part of his undergraduate thesis which was supervised by Professor Ore) extended Mills' result in the special case of odd perfect numbers with the aid of tables of Kraitchik [4, pp. 89, 91, 152–159] to obtain (4).

More recently, the help of digital computers was enlisted to prove the following:

Theorem. *Any odd perfect number must be divisible by a prime power greater than* 10^{12}.

Outline of Proof. Assume that every prime power factor of n is less than 10^{12}. Steuerwald showed that at least one of the b_i in equation (3) must be greater than 1 [10]. The corresponding q_i, therefore, must be less than 1000.

It was found that for each f^e, where f is a prime < 1000 and $f^e < 10^{12}$, eventually at least one of the following three contradictions develop by (repeated, if necessary) reference to (2):

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TABLE 1

Phase	Primes Eliminated											
1	127	271										
2	911											
3	19											
4	239	311	443	691	839	859						
5	179	919										
6	11											
7	163	467	619	857	883	971						
8	71	547										
9	587	593	709	1093								
10	151	227										
11	571											
12	109	461										
13	263	499	653									
14	7	359										
15	37	97	191	331	347	431	487	599	683	739	751	787
	823	863	907	977								
16	31	47	193	379	433	491	569	643	719	997		
17	61	281	293	349	557	631						
18	13	59	131									
19	23	113	167	233	337	353	367	389	419	503	523	607
	659	757	887									
20	3	137	229	283	373	677	733					
21	5	29	43	53	73	79	89	101	103	149	173	181
	199	223	241	257	269	307	317	383	401	439	449	457
	521	617	641	727	773	809	821	827	853	937	967	
22	17	41	83	157	211	251	397	409	479	509	541	601
	613	661	701	743	761	811	829	877	881	941	947	983
	991											
23	67	107	139	197	313	421	463	577	647	769	797	929
	953											
24	277	563	673									

(a) The integer n has a prime factor $F \equiv 3 \pmod 4$, $F > 10^6$. F has an even exponent by (3). But then $F^2 > 10^{12}$.

(b) A sequence of prime divisors develops that includes primes G, $H \equiv 1 \pmod 4$, where G is assigned an odd exponent and $H > 10^6$. By (3), H must have an even exponent, and $H^2 > 10^{12}$.

(c) A prime factor < 1000 (or 1093, which is specially included for convenience) that has been eliminated previously is encountered.

The proof was divided into twenty-four phases. A prime factor $f < 1000$ (or 1093) is eliminated during phase $P + 1$ if the previously eliminated primes upon which its exclusion depends include at least one prime in phase P. In order to shorten the proof, exclusions which depended upon previously eliminated primes were sought.

The 168 possible primes are eliminated successively in the order indicated in Table 1.

For reasons of space, only the first two phases of the proof are included here as Table 2. (The author will supply a copy of the complete proof upon request.) A copy has been placed in the UMT file of this journal.

TABLE 2

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127
   127 (2) = 3 * 5419
     5419 (2) = 3 * 31 * 313 * 1009
 P     1009 = 2 * 5 * 101
         101 (2) = 10303
           10303 (2) = 3 * 5827 * 6073
             6073 (2) = 3 * 7 * 139 * 12637
               12637 (2) = 3 * 7 * 73 * 104179
                 104179 (2) = 3 * 73 * 103 * 481153
                   481153 (2) = 3 * 15199 * 5077279 /-/
         101 (4) = 5 * 31 * 491 * 1381
           1381 (2) = 3 * 7 * 13 * 6991
             6991 (2) = 3 * 16293691 /-/
       1009 (2) = 3 * 37 * 9181
 P       9181 = 2 * 4591
           4591 (2) = 3 * 127 * 55333
             55333 (2) = 3 * 367 * 2780923 /-/
         9181 (2) = 3 * 7 * 7 * 13 * 31 * 1423
           1423 (2) = 3 * 7 * 96493
 P           96493 = 2 * 48247
               48247 (2) = 3 * 775940419 /-/
             96493 (2) = 3 * 19 * 163350799 /-/
   127 (4) = 262209281
 P   262209281 = 2 * 3 * 3137 * 13931
       13931 (2) = 194086693 /+/
271
   271 (2) = 3 * 24571
     24571 (2) = 3 * 201252871 /-/
   271 (4) = 5 * 251 * 4313591 /-/
911
   911 (2) = 830833
 P   830833 = 2 * 127 // * 3271
     830833 (2) = 3 * 13 * 61 * 337 * 861001
 P     861001 = 2 * 151 * 2851
         2851 (2) = 3 * 7 * 67 * 5779
           5779 (2) = 3 * 7 * 409 * 3889
             3889 (2) = 3 * 7 * 7 * 102913
               102913 (2) = 3 * 79 * 337 * 132607
                 132607 (2) = 3 * 103 * 109 * 127 // * 4111
       861001 (2) = 3 * 61 * 18661 * 217081
 P       217081 = 2 * 108541
           108541 (2) = 3 * 3927085741 /+/
         217081 (2) = 3 * 7 * 2083 * 1077301
 P         1077301 = 2 * 538651
             538651 (2) = 3 * 13 * 17509 * 424903
               424903 (2) = 3 * 60180994771 /-/
   911 (4) = 5 * 11 * 701 * 17884211 /-/
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The proof was recorded on punched cards, so only a restricted set of characters was available. The second and third lines of the proof would appear in conventional notation as follows:

$$\sigma(127^2) = 3 \cdot 5419,$$

$$\sigma(5419^2) = 3 \cdot 31 \cdot 313 \cdot 1009.$$

The three criteria for exclusion, (a), (b), and (c), are marked by placing the symbols /−/, /+/, and /   /, respectively, after the prime.

For primes $\equiv 1 \pmod 4$, the only odd exponent which had to be considered was 1, as $\sigma(p)$ divides $\sigma(p^{2m+1})$. The prime with the odd exponent is preceded by the letter P.

With this result, Kanold's lower bound of 10^{20} for an odd perfect number can be raised. To produce a specific number as a bound, however, it is necessary to assemble various other restrictions upon odd perfect numbers. This is not being undertaken here, as M. Garcia has obtained (but not published) a yet higher bound.

The University of Pittsburgh's IBM 7070 and IBM 7090 digital computers were used to obtain prime factorizations and to check the accuracy and completeness of the proof. The author wishes to express his appreciation to the University of Pittsburgh's Computation and Data Processing Center for granting access to these computers. This facility is supported in part under National Science Foundation Grants G11309 and GP2310.

University of Pittsburgh
Pittsburgh, Pennsylvania

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Solutions of the Diophantine Equations $x^2 + y^2 = l^2, y^2 + z^2 = m^2, z^2 + x^2 = n^2$

By M. Lal and W. J. Blundon

Introduction. The solution of the system of three equations of the second degree in six unknowns i.e. $x^2 + y^2 = l^2$, $y^2 + z^2 = m^2$ and $z^2 + x^2 = n^2$ is a classical Diophantine problem [1, p. 112]. The geometrical significance of this problem is to find a rectangular parallelepiped whose edges and face diagonals are all rational integers. If x, y and z are relatively prime in pairs the above system has no solution; otherwise there are infinitely many solutions.

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