Complex Zeros of Cylinder Functions

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Abstract. All complex zeros of the cylinder functions $Y_n(z)$, $H_n^{(1)}(z)$ for $n = 0$ and $1, |z| < 158, |\arg z| \leqq \pi$ and some complex zeros small in modulus of $Y_n(z)$, $H_n^{(1)}(z)$ for $n = 2, 3, 4, 5, 15$ and $|\arg z| \leqq \pi$ are tabulated accurate to $10D$. The methods (being partly new) for computing these zeros are described.

1. Introduction. A large number of positive real zeros of cylinder functions for the orders $\nu = n, n + \frac{1}{2}, n \pm \frac{1}{3}$ $(n = 0, 1, 2, \cdots)$ have been tabulated [1]. However only a few complex zeros are known [2], [3], [4]. Most of the latter were computed by means of Newton's method and are given to a few decimals only. Here a number of real and complex zeros have been computed (see Abstract).

Three different methods of computation have been used which, however, form a unified procedure for an electronic digital computer. In the following a zero will be called large (small), if it is large (small) in modulus. The large zeros were computed by means of the well-known McMahon expansion, for which a new algorithm was obtained and used. The smaller zeros have been determined by inverse interpolation with infinitely many coinciding interpolation points where the higher derivatives of the cylinder functions are expressed by the function itself and its first derivative. The coefficients of this series turn out to depend only on the argument z and the order ν. The interpolation point z_0, that is to say, a first approximation to a zero ξ is obtained either by means of the McMahon expansion or, for the zeros with $|\text{Re}(z)| < \nu = n$, by using Olver's expansions. Since these are very complicated the first term is taken only. An upper bound for the truncation error of the inverse series was obtained and used. Cancellation and rounding errors are excluded by using triple precision arithmetic. Since only zeros of functions with integer order are considered, the following results are written down for integer order denoted by n although they are valid for general orders ν. $\bar{a}$ denotes the conjugate of a, $[m]$ the integral part of m $(\geqq 0)$.

2. Types of Zeros. The complex zeros in question lie on the dotted lines of Figs. 1 and 2. There are two different types of zeros (for $|\arg z| \leqq \pi$):

1) An infinite number of zeros for $|\text{Re}(z)| > n$ which lie on the real axis or very close to it.

2) A group of n or $2n$ complex zeros for $|\text{Re}(z)| < n$ which lie along the boundary of an eye-shaped domain around $z = 0$. The notation of the zeros is as indicated in Figs. 1, 2. s denotes the number of the zero. The types $z_{n,s}^{Y}$, $z_{n,s}^{H1}$ are ordered according to increasing magnitude beginning with $s = 1$. The n zeros $z_{n,s}^{YU}$ and the n zeros $z_{n,s}^{H1U}$ are ordered in the sense of the arrows in Figs. 1 and 2. (The U in the notation $z_{n,s}^{YU}$ and $z_{n,s}^{H1U}$ was chosen to indicate that these zeros are, contrary to the others, irregularly spaced.)

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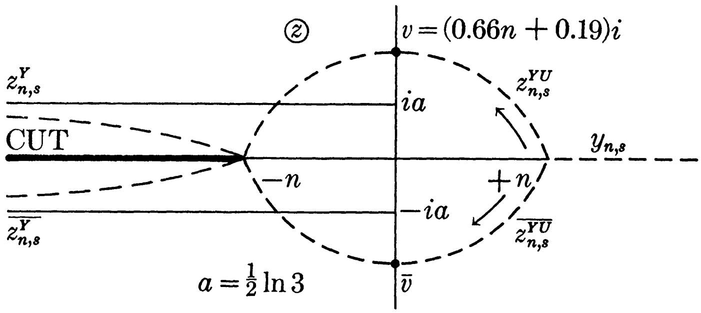

FIGURE 1

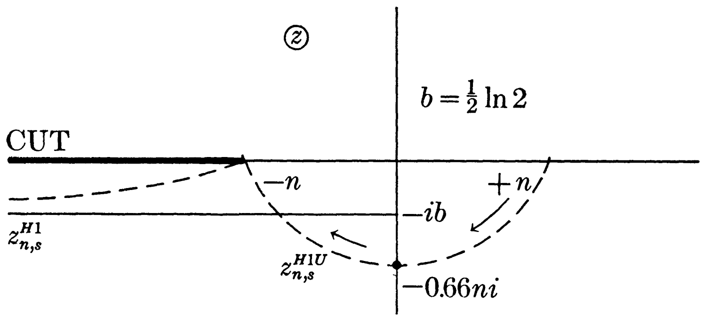

FIGURE 2

3. The McMahon Expansion. The large zeros have been computed by means of the McMahon expansion:

$$z_s \sim \beta - \sum_{k=0}^{\infty} \frac{D_{2k+1}}{\beta^{2k+1}}; \qquad \beta = \left(s + \frac{n}{2} - \frac{1}{4}\right)\pi - \alpha \tag{3.1}$$

where z_s stands for any of the larger zeros $-z^Y_{n,s}$, or $-z^{H1}_{n,s}$. The parameter α depends on the type of the zero and is given in Fig. 3. Following Watson's derivation of the McMahon expansion [5, pp. 504–507] an algorithm, called MCALG, for the calculation of the terms in this series has been obtained [6]:

1. $t_{k+1} = -\left(k + \frac{1}{2}\right) \cdot \frac{n^2 - (k + \frac{1}{2})^2}{k+1} \cdot t_k$; $t_0 = 1$

2. $A'_{2k+2} = -\sum_{\lambda=0}^{k} (-1)^{k-\lambda+1} A'_{2\lambda} \cdot t_{k-\lambda+1}$; $A_0' = 1$

3. $C_{2k+1} = -\frac{A'_{2k+2}}{2k+1}$

4. $Q_k = \sum_{\lambda=0}^{k} C_{2(k-\lambda)-1} \cdot C_{2\lambda-1}$; $C_{-1} = 1$

5. $E^{(2\kappa+1)}_{2k+1} = E^{(2\kappa-1)}_{2k-1} - \sum_{\lambda=1}^{k-\kappa} Q_\lambda \cdot E^{(2\kappa+1)}_{2(k-\lambda)+1}$; $\kappa = 0, 1, \cdots, k$; $E^{(-1)}_{2k-1} = C_{2k-1}$

6. $D_{2k+1} = C_{2k+1} + \sum_{\lambda=1}^{k} D_{2\lambda-1} \cdot E^{(2\lambda-1)}_{2k+1}$.

For 1. to 6.: $k = 0, 1, 2, \cdots$.

Type of zero	$z^{Y}_{n,s}$	$z^{H1}_{n,s}$
Parameter α	$-\frac{i}{2}\ln 3$	$\frac{i}{2}\ln 2$

FIGURE 3

Type of zero	s	W	g_s
$z^{YU}_{n,s}$	$1, 2, \cdots, \left[\frac{n}{2}\right]$	$n\cdot\overline{t(g_s)}$	β_s
$z^{YU}_{n,n-s+1}$	$1, 2, \cdots, \left[\frac{n+1}{2}\right]$	$-n\cdot t(g_s)$	δ_s
$z^{H1}_{n,s}$	$1, 2, \cdots, \left[\frac{n}{2}\right]$	$n\cdot t(g_s)$	$a_s\cdot e^{-2\pi i/3}$
$z^{H1U}_{n,n-s+1}$	$1, 2, \cdots, \left[\frac{n+1}{2}\right]$	$-n\cdot t(g_s)$	$b_s\cdot e^{2\pi i/3}$

FIGURE 4

4. The Olver Expansion. The smaller zeros have been computed by using the method of inverse interpolation described in the next section. There an approximate value z_0 for a zero is needed. For the zeros $z^{YU}_{n,s}$ and $z^{H1U}_{n,s}$ the value z_0 is obtained by means of Olver's asymptotic expansion [1], [2]. This is an expansion in $1/n$ uniformly valid in z. The terms decrease rapidly even for small values of n. However, the coefficients are very complicated. Therefore, only the first term W has been used to compute z_0, thus $z_0 = W$ and

$$\xi = W + O(1/n). \tag{4.1}$$

ξ is any zero of a general cylinder function $C_n(z)$. W may be taken from Fig. 4. g_s is a corresponding zero of an Airy function, i.e. of a solution of Airy's differential equation

$$w'' = zw.$$

"Corresponding" means: the integer s in the first column of Fig. 4 is equal to that in the last column. The a_s are the negative zeros of $Ai(z)$, the b_s those of $Bi(z)$. β_s are the zeros of $Bi(z)$ nearby the ray $\arg z = \pi/3$. δ_s are the zeros of $Di(z) = 2iAi(z) - Bi(z)$ nearby the ray $\arg z = \pi/3$.

The small zeros of the Airy functions can be determined by a series of the type mentioned in the next section with similar coefficients [7], the larger ones by the McMahon expansion. It can be shown that (3.1) with MCALG can be used here too; only the first step has to be replaced by (see [6]).

$$1.\ t_{k+1} = \frac{(6k+1)(6k+3)(6k+5)}{216(k+1)} \cdot t_k;\ t_0 = 1.$$

The relation $t(g_s)$ is given by

$$t = \frac{1}{\cosh\sigma};\qquad \sigma - \tanh\sigma = \frac{2}{3}\cdot\frac{1}{n}\cdot g_s^{3/2} \equiv \rho \tag{4.2}$$

$$(\arg\rho = \tfrac{3}{2}\arg g_s;\ -\pi < \arg g_s \leqq \pi;\ \sigma > 0 \text{ for } \rho > 0).$$

TABLE I

All Zeros of $H_0^{(1)}(z)$ and $H_1^{(1)}(z)$ for $|z| < 158$ and $|\arg z| \leqq \pi$

s	The zeros of $H_0^{(1)}(z)$		The zeros of $H_1^{(1)}(z)$	
	Real part	Imaginary part	Real part	Imaginary part
—	—	—	−0,41927 46041	−0,57739 95241
1	−2,40409 11772	−0,34050 21530	−3,83244 28677	−0,35490 47062
2	−5,51999 75208	−0,34522 50285	−7,01571 33208	−0,34916 14583
3	−8,65370 57658	−0,34600 81928	−10,17351 02841	−0,34781 69367
4	−11,79152 55075	−0,34626 59215	−13,32371 08126	−0,34730 14370
5	−14,93091 32687	−0,34638 08037	−16,47064 00729	−0,34705 08281
6	−18,07106 14513	−0,34644 16537	−19,61586 44534	−0,34691 04346
7	−21,21163 50691	−0,34647 76880	−22,76008 81891	−0,34682 39696
8	−24,35247 04974	−0,34650 07611	−25,90367 46728	−0,34676 69754
9	−27,49347 84130	−0,34651 64143	−29,04683 03693	−0,34672 74366
10	−30,63460 59482	−0,34652 75171	−32,18968 12593	−0,34669 88905
11	−33,77581 98251	−0,34653 56755	−35,33230 85700	−0,34667 76100
12	−36,91709 80560	−0,34654 18452	−38,47476 70248	−0,34666 13237
13	−40,05842 55316	−0,34654 66235	−41,61709 48371	−0,34664 85829
14	−43,19979 15273	−0,34655 03991	−44,75931 94996	−0,34663 84285
15	−46,34118 82210	−0,34655 34342	−47,90146 12967	−0,34663 02050
16	−49,48260 97737	−0,34655 59104	−51,04353 55220	−0,34662 34522
17	−52,62405 17382	−0,34655 79569	−54,18555 39240	−0,34661 78393
18	−55,76551 06685	−0,34655 96676	−57,32752 56769	−0,34661 31234
19	−58,90698 38527	−0,34656 11123	−60,46945 80490	−0,34660 91231
20	−62,04846 91274	−0,34656 23432	−63,61135 68734	−0,34660 57006
21	−65,18996 47461	−0,34656 34006	−66,75322 68855	−0,34660 27497
22	−68,33146 92828	−0,34656 43157	−69,89507 19694	−0,34660 01877
23	−71,47298 15625	−0,34656 51128	−73,03689 53411	−0,34659 79491
24	−74,61450 06076	−0,34656 58114	−76,17869 96865	−0,34659 59816
25	−77,75602 55985	−0,34656 64271	−79,32048 72657	−0,34659 42433
26	−80,89755 58428	−0,34656 69725	−82,46225 99947	−0,34659 26998
27	−84,03909 07517	−0,34656 74579	−85,60401 95081	−0,34659 13230
28	−87,18062 98210	−0,34656 78918	−88,74576 72094	−0,34659 00899
29	−90,32217 26168	−0,34656 82812	−91,88750 43097	−0,34658 89810
30	−93,46371 87636	−0,34656 86320	−95,02923 18605	−0,34658 79803
31	−96,60526 79343	−0,34656 89492	−98,17095 07784	−0,34658 70741
32	−99,74681 98436	−0,34656 92369	−101,31266 18663	−0,34658 62509
33	−102,88837 42404	−0,34656 94986	−104,45436 58308	−0,34658 55008
34	−106,02993 09039	−0,34656 97375	−107,59606 32957	−0,34658 48155
35	−109,17148 96383	−0,34656 99560	−110,73775 48141	−0,34658 41876
36	−112,31305 02699	−0,34657 01564	−113,87944 08781	−0,34658 36110
37	−115,45461 26439	−0,34657 03407	−117,02112 19270	−0,34658 30802
38	−118,59617 66219	−0,34657 05106	−120,16279 83541	−0,34658 25905
39	−121,73774 20796	−0,34657 06675	−123,30447 05127	−0,34658 21378
40	−124,87930 89055	−0,34657 08127	−126,44613 87208	−0,34658 17183
41	−128,02087 69989	−0,34657 09473	−129,58780 32658	−0,34658 13290
42	−131,16244 62686	−0,34657 10724	−132,72946 44078	−0,34658 09670
43	−134,30401 66321	−0,34657 11888	−135,87112 23827	−0,34658 06299
44	−137,44558 80145	−0,34657 12973	−139,01277 74054	−0,34658 03153
45	−140,58716 03475	−0,34657 13986	−142,15442 96715	−0,34658 00213
46	−143,72873 35686	−0,34657 14934	−145,29607 93599	−0,34657 97463
47	−146,87030 76211	−0,34657 15821	−148,43772 66341	−0,34657 94885
48	−150,01188 24525	−0,34657 16653	−151,57937 16443	−0,34657 92465
49	−153,15345 80150	−0,34657 17435	−154,72101 45284	−0,34657 90191
50	−156,29503 42646	−0,34657 18170	−157,86265 54134	−0,34657 88052

TABLE II

All Complex Zeros of $Y_0(z)$ *and* $Y_1(z)$ *for* $|z| < 158$ *and* $|\arg z| \leqq \pi$*

s	The zeros of $Y_0(z)$		The zeros of $Y_1(z)$	
	Real part	Imaginary part	Real part	Imaginary part
—	—	—	−0,50274 32730	0,78624 37145
1	−2,40301 66320	0,53988 23130	−3,83353 51926	0,56235 65382
2	−5,51987 67024	0,54718 00106	−7,01590 36832	0,55339 30459
3	−8,65367 24031	0,54841 20673	−10,17357 38345	0,55127 33877
4	−11,79151 20304	0,54881 91184	−13,32373 93075	0,55045 85830
5	−14,93090 65640	0,54900 08289	−16,47065 52105	0,55006 20446
6	−18,07105 76493	0,54909 71445	−19,61587 34327	0,54983 97790
7	−21,21163 27107	0,54915 42036	−22,76009 39447	0,54970 28473
8	−24,35246 89357	0,54919 07480	−25,90367 85802	0,54961 25698
9	−27,49347 73262	0,54921 55442	−29,04683 31422	0,54954 99326
10	−30,63460 51618	0,54923 31341	−32,18968 32975	0,54950 47056
11	−33,77581 92380	0,54924 60604	−35,33231 01117	0,54947 09878
12	−36,91709 76061	0,54925 58363	−38,47476 82191	0,54944 51815
13	−40,05842 51793	0,54926 34078	−41,61709 57810	0,54942 49924
14	−43,19979 12463	0,54926 93908	−44,75932 02584	0,54940 89011
15	−46,34118 79933	0,54927 42004	−47,90146 19158	0,54939 58694
16	−49,48260 95866	0,54927 81244	−51,04353 60338	0,54938 51681
17	−52,62405 15827	0,54928 13676	−54,18555 43519	0,54937 62730
18	−55,76551 05378	0,54928 40787	−57,32752 60382	0,54936 87993
19	−58,90698 37418	0,54928 63682	−60,46945 83568	0,54936 24596
20	−62,04846 90325	0,54928 83190	−63,61135 71379	0,54935 70356
21	−65,18996 46642	0,54928 99948	−66,75322 71143	0,54935 23591
22	−68,33146 92117	0,54929 14450	−69,89507 21687	0,54934 82987
23	−71,47298 15004	0,54929 27083	−73,03689 55159	0,54934 47508
24	−74,61450 05530	0,54929 38155	−76,17869 98405	0,54934 16327
25	−77,75602 55502	0,54929 47913	−79,32048 74021	0,54933 88776
26	−80,89755 57999	0,54929 56556	−82,46226 01161	0,54933 64313
27	−84,03909 07134	0,54929 64249	−85,60401 96167	0,54933 42494
28	−87,18062 97867	0,54929 71126	−88,74576 73068	0,54933 22950
29	−90,32217 25860	0,54929 77298	−91,88750 43975	0,54933 05376
30	−93,46371 87358	0,54929 82858	−95,02923 19399	0,54932 89515
31	−96,60526 79092	0,54929 87885	−98,17095 08504	0,54932 75153
32	−99,74681 98207	0,54929 92445	−101,31266 19318	0,54932 62106
33	−102,88837 42196	0,54929 96593	−104,45436 58906	0,54932 50218
34	−106,02993 08848	0,54930 00379	−107,59606 33503	0,54932 39356
35	−109,17148 96208	0,54930 03842	−110,73775 48642	0,54932 29405
36	−112,31305 02539	0,54930 07019	−113,87944 09242	0,54932 20266
37	−115,45461 26292	0,54930 09940	−117,02112 19695	0,54932 11854
38	−118,59617 66083	0,54930 12632	−120,16279 83934	0,54932 04092
39	−121,73774 20671	0,54930 15118	−123,30447 05490	0,54931 96916
40	−124,87930 88939	0,54930 17419	−126,44613 87545	0,54931 90269
41	−128,02087 69880	0,54930 19553	−129,58780 32971	0,54931 84098
42	−131,16244 62585	0,54930 21536	−132,72946 44369	0,54931 78361
43	−134,30401 66227	0,54930 23381	−135,87112 24099	0,54931 73017
44	−137,44558 80058	0,54930 25101	−139,01277 74308	0,54931 68031
45	−140,58716 03393	0,54930 26707	−142,15442 96952	0,54931 63372
46	−143,72873 35610	0,54930 28208	−145,29607 93821	0,54931 59012
47	−146,87030 76139	0,54930 29615	−148,43772 66549	0,54931 54926
48	−150,01188 24458	0,54930 30934	−151,57937 16639	0,54931 51092
49	−153,15345 80087	0,54930 32173	−154,72101 45468	0,54931 47488
50	−156,29503 42587	0,54930 33337	−157,86265 54307	0,54931 44098

* The conjugate complex values of the zeros given in Table III are also zeros of $Y_0(z)$ and $Y_1(z)$, respectively, in the z-domain mentioned above.

TABLE III

Some Small Complex Zeros of $H_n^{(1)}(z)$ and $Y_n(z)$ for $|\arg z| \leqq \pi, n = 2, 3, 4, 5, 15$

n	s	Zeros of $H_n^{(1)}(z)$* Real part	Zeros of $H_n^{(1)}(z)$* Imaginary part	Zeros of $Y_n(z)$* Real part	Zeros of $Y_n(z)$* Imaginary part
2	1	0.42948 49652	−1.28137 37977	0.47212 88545	1.48166 54964
	2	−1.31684 11674	−0.83610 44833	−1.47994 58229	1.06259 06235
	1	−5.13755 90975	−0.37221 27527	−5.14043 97188	0.58957 98497
3	1	1.30801 20323	−1.68178 88047	1.40320 35309	1.88769 05713
	2	−0.43182 10011	−1.95858 45276	−0.46046 54313	2.15604 55555
	3	−2.24246 92551	−1.00648 23832	−2.45453 50853	1.24889 89660
	1	−6.38302 25023	−0.38848 59998	−6.38728 35322	0.61523 04718
4	1	2.20437 19815	−1.97816 18635	2.33510 20228	2.19068 52105
	2	0.43269 66486	−2.62867 11680	0.45425 93026	2.82468 94115
	3	−1.30516 66262	−2.42571 74415	−1.37362 79706	2.62593 42089
	4	−3.18191 42414	−1.13815 52861	−3.33074 35178	1.39457 03562
	1	−7.59195 14207	−0.40346 11183	−7.59733 17993	0.63885 75932
5	1	3.11308 29450	−2.21862 62746	3.27138 80388	2.43769 82981
	2	1.30388 23977	−3.13513 28447	1.35764 92491	3.33278 81888
15	5	4.80157 02279	−9.12050 74552	4.86947 64115	9.31727 41444

* The zeros tabulated are $z^{H1U}_{n,s}$ and $z^{YU}_{n,s}$, resp., except for the zeros in the last lines of $n = 2, 3, 4$ which are $z^{H1}_{n,s}$ and $z^{Y}_{n,s}$, resp. (cp. Figs. 1 and 2).

The transcendental equation may be solved by a third order iterative method:

$$(4.3)\qquad \sigma_{k+1} = \sigma_k - \left[1 + \left(\frac{1}{\tanh \sigma_k} - \tanh \sigma_k\right) Q_k\right] Q_k; \qquad Q_k = \frac{\sigma_k - \tanh \sigma_k - \rho}{\tanh^2 \sigma_k}$$

with the initial values

$$(4.4)\qquad \begin{aligned} \sigma_0 &= 1.1 + \frac{\pi i}{4} && \text{for the zeros with } \frac{\pi}{3} \leqq \arg g_s < \frac{\pi}{2}, \\ \sigma_0 &= 1.1 - \frac{\pi i}{4} && \text{for the zeros with } -\frac{\pi}{2} < \arg g_s \leqq -\frac{\pi}{3}. \end{aligned}$$

The method converges near a zero of $f(\sigma) = \sigma - \tanh \sigma - \rho$, since there $f'(\sigma) \neq 0$ and $f^{(k)}(\sigma)$ $(k = 1, 2, 3)$ has no singularities there. The method (4.3) together with (4.4) converges fairly good for the zeros computed here.

In order to give an idea of the rate of convergence of (4.3) and to show how useful even the first term of Olver's expansion is, and that even for a small value of n, an example may be given.

Example. Computation of a first approximation $z_0 = W$ for $z^{YU}_{3,1}$. From the McMahon expansion (cp. [2], Table 1)

$$\beta_1 = 2.354 \exp\left(\frac{\pi}{3} + 0.095\right) i.$$

With (4.2)

$$\rho = -0.113984 + 0.794468 \cdot i; \qquad \sigma_0 = 1.1 + \frac{\pi i}{4}.$$

In the following the step by step corrections according to (4.3) are written down (all computations accurate to $10D$):

$$\begin{array}{rl}
\sigma = & 1.10000\ 00000 + 0.78539\ 81633i \\
& -0.\ 8618\ 67365 + 0.27240\ 06344i \\
& -0.\ 1095\ 50708 - 0.\ \ 116\ 29810i \\
& +0.\ \qquad 4644 - 0.\ \qquad 1163i \\
& +0.\ \qquad\quad 0 + 0.\ \qquad\quad 0i \\
\hline
& +1.00285\ 86571 + 1.05663\ 59330i
\end{array}$$

According to (4.2):

$$t = 0.46563\ 21311 - 0.62882\ 32811i.$$

As to Fig. 4: $z_{3,1}^{YU} = 3 \cdot \overline{t(g_s)} + O(n^{-1}) = 1.39689\ 63933 + 1.88646\ 98433i + O(n^{-1})$. According to Table III accurate to $10D$: $z_{3,1}^{YU} = 1.40320\ 35309 + 1.88769\ 05713i$. The first approximation is thus, even for $n = 3$, correct to about 0.5%.

5. Inverse Series. The smaller zeros were computed by means of the inverse series for a zero ξ of any cylinder function $C_n(z)$

$$\xi = z_0 - \sum_{\lambda=1}^{\infty} (-1)^{\lambda} \frac{D_{\lambda-1}}{\lambda!} h^{\lambda} \tag{5.1}$$

with

$$-\frac{1}{h} = \frac{C_n'(z_0)}{C_n(z_0)} = \frac{n}{z_0} - \frac{C_{n+1}(z_0)}{C_n(z_0)}$$

where z_0 is an approximation to ξ and $z_0^{\lambda} D_{\lambda}$ are polynomials in z_0^2, the coefficients of which are again polynomials in n^2. This series was obtained from the Taylor series of the inverse function of a certain function that has, in a certain domain of the z-plane, the same zeros as $C_n(z)$. Formulae for the D_{λ}, a sufficient condition for the convergence of the above series, a local existence statement, an upper bound for the truncation error and numerical discussions of the rate of convergence are given in [6]. (In [8] these results are given for the more general case of functions satisfying a second order differential equation.) Miller and Jones [9] have given a similar series as (5.1) which is a power series in $C_n(z_0)/C_{n-1}(z_0)$, but they have not considered the convergence of their expansion.

6. Accuracy. The numerical computations were carried out on an IBM 704 of Deutsches Rechenzentrum at Darmstadt. The zeros tabulated are supposed to be accurate to $< 5 \cdot 10^{-11}$ in absolute value, i.e. to 10D.

7. Tables. In the Tables I to III the zeros mentioned in the abstract are tabulated.

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