The Number of Lattice Points in a k-dimensional Hypersphere*

By W. C. Mitchell

1. Introduction. One of the most interesting problems of analytic number theory involves the difference between the number of lattice points in a k-dimensional hypersphere and the "volume" of the hypersphere. Define the set $L_k(x)$ as follows:

$$L_k(x) = \left\{ (J_1, J_2, \cdots, J_k) \;\middle|\; \sum_{i=1}^{k} J_i^2 \leqq x \right\} \tag{1}$$

where the J_i are integers. Let $A_k(x)$ be the number of distinct points in $L_k(x)$. Thus $A_k(x)$ is the number of lattice points in a k-dimensional hypersphere of radius $x^{1/2}$. Define $V_k(x)$ as the "volume" of a k-dimensional hypersphere of radius $x^{1/2}$.

$$V_k(x) = \frac{\pi^{k/2} x^{k/2}}{\Gamma\left(\frac{k}{2}+1\right)} = \frac{\pi^{[k/2]} x^{k/2}}{\frac{k}{2}\left(\frac{k}{2}-1\right)\cdots\left(1 \text{ or } \frac{1}{2}\right)} \tag{2}$$

where $[z]$ is the integer part of z.

The problem of primary interest is to find the Greatest Lower Bound θ_k of the set of values θ for which

$$P_k(x) \equiv A_k(x) - V_k(x) = O(x^\theta). \tag{3}$$

Walfisz [1] gives the following general results:

$$\begin{aligned} P_k(x) &= O(x^{(k-1)/2}), \quad P_k(x) = \Omega(x^{k/2-1}), \\ P_4(x) &= O(x \log^2 x) = O(x^{1+\epsilon}), \qquad \epsilon > 0, \\ P_k(x) &= O(x^{k/2-1}), \qquad k \geqq 5. \end{aligned} \tag{4}$$

Thus for $k \geqq 4$ $\theta_k = k/2 - 1$.

The value of k which has received the greatest attention is $k = 2$, the number of lattice points in a circle. Wilton [2] gives an account of the early work in this problem. Since that time several results have been published establishing new values of θ for which $P_2(x) = O(x^\theta)$. One of the most recent is Chen Jing-ren's proof [3] that $P_2(x) = O(x^{12/37})$. Hardy (see [2]) has shown that $P_2(x) = \Omega(x^{1/4})$. It is a common conjecture that $P_2(x) = O(x^{1/4+\epsilon})$, $\epsilon > 0$, or $\theta_2 = \frac{1}{4}$.

There is less known for $k = 3$. From (4) we have $\frac{1}{2} \leqq \theta_3 \leqq 1$. Fraser and Gotlieb [4] conjectured on the basis of numerical evidence that $.5 \leqq \theta_3 \leqq .7$. More recently Chen Jing-ren [5] has shown that $\frac{1}{2} \leqq \theta_3 \leqq \frac{2}{3}$.

With the advent of high speed computers it has become possible to evaluate

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$P_k(x)$ for "large" x in order to see if the calculated results are consistent with theoretical results or if it is reasonable to make any new conjectures concerning θ_k. There have been at least three previous papers on this subject. Fraser and Gotlieb [4] calculated isolated values of $P_2(x)$ and $P_3(x)$ for $x^{1/2} < 2000$ on an IBM 650. However their conclusions differ with the present paper for θ_2. Harry Mitchell [6] calculated $P_2(x)$ for $x^{1/2} \leqq 200{,}000$ on an IBM 7090, but the results for $x^{1/2} \geqq 3000$ are incorrect, as pointed out by Keller and Swenson [7]. Keller and Swenson determined $P_2(x)$ for many values of $x^{1/2} < 260{,}000$ on an IBM 7090 and their method of interpretation leads them to suggest that $\theta_2 \leqq .3$. This seems unlikely from the results of the present method of interpretation.

The problem of establishing that $P_k(x) = O(x^\theta)$ is equivalent to finding a sequence $\langle (X_i, Y_i)\rangle_{i=1}^\infty$ such that

$$(5) \qquad |P_k(x)| \leqq Y_i + O(x^\theta) \quad \text{for} \quad x \leqq X_{i+1} \quad \text{and} \quad \limsup_{i\to\infty} \frac{Y_i}{X_i^\theta} < \infty.$$

Since $L_k(x)$ is composed only of k-tuples of integers, $A_k(x)$ is piecewise constant over $[n, n+1)$, where n is an integer. Thus for $x \in [n, n+1)$

$$(6) \qquad \operatorname*{Lim}_{\beta\to 1^-} P_k(n+\beta) \leqq P_k(x) \leqq P_k(n).$$

But

$$\begin{aligned} \operatorname*{Lim}_{\beta\to 1^-} P_k(n+\beta) &= A_k(n) - V_k(n+1) \\ (7) \qquad &= P_k(n) - (V_k(n+1) - V_k(n)) \\ &= P_k(n) + O(x^{k/2-1}). \end{aligned}$$

However, by (4), $\theta_k \geqq k/2 - 1$. Therefore the sequence of "extreme" points $(N_i, |P_k(N_i)|)$, defined such that $|P_k(n)| < |P_k(N_i)|$ for $n < N_{i+1}$, satisfies the first requirement of (5) for all θ of interest. This sequence is uniquely determined, given an initial element, and N_{i+1} is the first integer for which $|P_k(N_i)| < |P_k(N_{i+1})|$.

For x too large to calculate $P_k(x)$ conveniently for all integers, approximate extreme points can be chosen in the same manner as the true extreme points but from a more restricted set. These approximate extreme points are not necessarily a subset of the true extreme points. This later method, used by both Fraser and Gotlieb and Keller and Swenson (in a different context), is not so concise as the former but allows one to consider a larger range of x.

The present calculations on an IBM 7094 include $P_k(x)$ for $k = 2, 3, 4, 5, 6$ and all integer $x \leqq 250{,}000$ ($x^{1/2} \leqq 500$); some 250 isolated values of $P_2(x)$ for $x^{1/2} \leqq 10{,}000{,}000$; and about 20 values of $P_3(x)$ for $x^{1/2} \leqq 9000$. The results of this work show that the calculated values follow the theoretical limits quite closely. The results for $k = 2$ fail to indicate that θ_2 is less than Chen Jing-ren's bound of $12/37 = \overline{.324}$. For $k = 3$ the most reasonable conclusion is $.5 \leqq \theta_3 \leqq .6$.

Efficient algorithms for various combinations of k and x are presented in Section 2. Section 3 is composed of computing methods and Section 4 contains conclusions.

2. Counting Algorithms. The most efficient method of evaluating $A_k(x)$ depends on the range of k and x and upon whether isolated values (for approximate extreme points) or a large number of consecutive values (for true extreme points) is desired. The following formula is similar to one given by Walfisz [1].

$$\begin{aligned} A_1(0) &= 1, \\ A_1(x) &= A_1(x-1) + 2\delta(x) \end{aligned} \tag{8}$$

where

$$\delta(x) = \begin{cases} 1, & \text{for } x \text{ a perfect square,} \\ 0, & \text{otherwise,} \end{cases}$$

$$A_k(x) = A_{k-1}(x) + 2\sum_{i=1}^{[\sqrt{x}]} A_{k-1}(x - i^2).$$

Formula (8) provides the basic method of calculating $A_k(x)$. For large values of x, some terms of the above summation may be larger than the fixed-point single-word capacity of the computer ($2^{36} - 1$ on the IBM 7094). This difficulty can often be remedied by defining $R_k(x)$ as the number of points $(J_1, J_2, \cdots, J_k)$ such that

$$\sum_{i=1}^{k} J_i^2 = x. \tag{9}$$

It is evident that

$$\begin{aligned} R_k(x) &= \begin{cases} A_k(x) - A_k(x-1), & x \text{ an integer,} \\ 0, & \text{otherwise,} \end{cases} \\ A_k(x) &= \sum_{i=0}^{[x]} R_k(i). \end{aligned} \tag{10}$$

Also

$$\begin{aligned} R_1(0) &= 1, \\ R_1(x) &= 2\delta(x), \\ R_k(x) &= R_{k-1}(x) + 2\sum_{i=1}^{[\sqrt{x}]} R_{k-1}(x - i^2). \end{aligned} \tag{11}$$

The similarities of (8) and (11) are noticeable. By changing initial values the same procedure may be used for either $A_k(x)$ or $R_k(x)$.

The next formula makes use of the symmetries involved in the set $L_k(x)$ resulting from permutations and negatives of ordered k-tuples. Define

$$L_k'(x) = \{(J_1, J_2, \cdots, J_k) \in L_k(x) \mid 0 \leqq J_1 \leqq J_2 \leqq \cdots \leqq J_k\} \tag{12}$$

and let $M(J_1, J_2, \cdots, J_k)$ be the number of distinct permutations and negations of $J_1, J_2, \cdots, J_k$. Then we have

$$M(J_1, J_2, \cdots, J_k) = \frac{2^{k-n(0)}k!}{\prod_{p=0}^{J_k} n(p)!}. \tag{13}$$

where $n(p)$ is the number of i for which $J_i = p$. Thus, if $\mathbf{Y_m} = (J_1, J_2, \cdots J_m)$,

$$(14) \qquad A_k(x) = \sum_{\mathbf{Y_k} \in L_k(x)} 1 = \sum_{\mathbf{Y_k} \in L_k'(x)} M(\mathbf{Y_k}).$$

By rearranging (12) and (14) it follows that

$$(15) \qquad A_k(x) = 1 + \sum_{J_k=1}^{[\sqrt{x}]} \sum_{\mathbf{Y_{k-1}} \in L_{k-1}'(x-J_k^2); J_{k-1} \leqq J_k} M(\mathbf{Y_{k-1}}, J_k).$$

Now, if $J_{k-1} < J_k$, then $M(J_1, J_2, \cdots, J_k) = 2kM(J_1, J_2, \cdots, J_{k-1})$. Similarly, if $J_{k-i} < J_{k-i+1} = \cdots = J_k$, then

$$M(J_1, J_2, \cdots, J_k) = 2^i \binom{k}{i} M(J_1, J_2, \cdots, J_{k-i}).$$

Thus we have

$$(16) \qquad A_k(x) = 1 + \sum_{J_k=1}^{[\sqrt{x}]} \sum_{i=1}^{k} 2^i \binom{k}{i} \sum_{\mathbf{Y_{k-i}} \in L_{k-i}(x-iJ_k^2); J_{k-i} < J_k} M(\mathbf{Y_{k-i}}).$$

Now define

$$S_m(Z, J) = \sum_{\mathbf{Y_m} \in L_m'(Z); J_m < J} M(\mathbf{Y_m}).$$

Thus

$$(17) \qquad A_k(x) = 1 + \sum_{J=1}^{[\sqrt{x}]} \sum_{i=1}^{k} 2^i \binom{k}{i} S_{k-i}(x - iJ^2, J).$$

$S_m(Z, J)$ can be defined recursively as follows:

$$(18) \qquad \begin{aligned} S_m(Z, J) &= 1 + \sum_{J_m=1}^{J-1} \sum_{i=1}^{m} 2^i \binom{m}{i} S_{m-i}(Z - iJ_m^2, J_m), \\ S_0(Z, J) &= \begin{cases} 1, & Z \geqq 0, \\ 0, & Z < 0. \end{cases} \end{aligned}$$

It is convenient to note that

$$(19) \qquad \begin{aligned} S_k(x, \infty) &= A_k(x), \\ S_k(\infty, J) &= (2J - 1)^k. \end{aligned}$$

Formula (18) used with (19) is the basic method for taking advantage of symmetries among the points of $L_k(x)$. By algebraic reduction the following general formula can be established:

$$(20) \qquad S_m(Z, J) = (2N + 1)^m + \sum_{i=1}^{m-1} \sum_{J_m=N+1}^{\mathrm{MIN}_i} 2^i \binom{m}{i} S_{m-i}(Z - iJ_m^2, J_m)$$

where $N = [\sqrt{Z/m}]$ and $\mathrm{MIN}_i = \min([\sqrt{Z/i}], J - 1)$. For complete generality $A_k(x)$ must be defined as in (19). For $k = 2$ this simplifies to

$$(21) \qquad A_2(x) = 1 + 4[\sqrt{x}] + 4[\sqrt{x/2}]^2 + 8 \sum_{J=[\sqrt{x/2}]+1}^{[\sqrt{x}]} [\sqrt{x - J^2}].$$

This formula was known to Gauss.

Formula (20) is of course ideally suited to programming for an algorithmic compiler; however, it can easily and efficiently be programmed in a machine-oriented language. Consequently it was coded in SCATRE for the 7094 and used for isolated values of $A_k(x)$ for large x.

One extension of (21) should be very valuable for computing isolated values of $A_2(x)$. It is possible, for certain x, to compute $A_2(x)$ for all u subject to $|u| < 2\sqrt{2}\, x^{1/4}$ in nearly the same time necessary to compute $A_2(x+u)$ alone. For $x = 10^{14}$, the largest argument used in this work, this would have made available over 17,000 results in about twice the time required for the single value. Needless to say, this is a significant improvement. Unfortunately, this method was not known when the computations were done for this paper, but the method has been used for moderate x ($x \approx 10{,}000$).

Define the following:

$$\begin{aligned} x &= r^2 + 2r = (r+1)^2 - 1, \quad r \text{ an integer}, \\ U &\leqq 2\sqrt{2r}, \\ -U + 2 &\leqq u \leqq U, \\ W &= [\sqrt{x - J^2}], \quad \text{where } J \text{ is used in the context of (21).} \end{aligned} \tag{22}$$

The following theorem may be established by simple algebra:

$$[\sqrt{x+u-J^2}] = \begin{cases} W+1, & u \geqq (W+1)^2 - (x - J^2), \\ W-1, & u < W^2 - (x - J^2), \\ W, & \text{otherwise}. \end{cases} \tag{23}$$

Thus, if the remainder of the integer square root routine is available, it is easy to evaluate $[\sqrt{x+u-J^2}]$ in the process of applying (21).

The true value of this method lies in computing $A_2(x+u)$ for all suitable u simultaneously. Define

$$Q(0) = 0,$$
$$q(u) = 0, \quad -U + 2 \leqq u \leqq U.$$

As J runs from $[\sqrt{x/2}] + 1$ to $[\sqrt{x}] = r$, as in (21), do the following:

$$\begin{aligned} Q(0) &= q(0) + [\sqrt{x - J^2}] = Q(0) + W, \\ q(v) &= q(v) + 1, \quad \text{where} \quad v = (W+1)^2 - (x - J^2), \\ q(v) &= q(v) - 1, \quad \text{where} \quad v = W^2 - (x - J^2) - 1. \end{aligned} \tag{24}$$

Then, for $u > 0$,

$$\begin{aligned} Q(u) &= Q(0) + \sum_{v=1}^{u} q(v) + [\sqrt{u-1}] - \sum_{J=[\sqrt{x/2}]+1}^{[\sqrt{(x+u)/2}]} [\sqrt{x+u-J^2}], \\ Q(-u) &= Q(0) + \sum_{v=1}^{u} q(-v) + \sum_{J=[\sqrt{(x-u)/2}]+1}^{[\sqrt{x/2}]} [\sqrt{x-u-J^2}]. \end{aligned} \tag{25}$$

And then, for all u,

$$A_2(x+u) = 1 + 4[\sqrt{x+u}] + 4[\sqrt{(x+u)/2}]^2 + 8Q(u). \tag{26}$$

3. Computer Methods and Numerical Results. The time-consuming part of computing $A_k(x)$ by any of the methods mentioned in this paper is evaluating $[\sqrt{x}]$. However, from the logical order of summation successive arguments often happen to be close together. Furthermore, while most square root routines are floating-point, exact fixed-point results are necessary for this work. Thus the efficiency of the square root routine may be improved by using fixed-point operations and by using the previous result as a first approximation for the current argument. Using this and the identity

$$(N \pm 1)^2 = N^2 \pm 2N + 1,$$

$A_k(x)$ may be calculated on a binary machine with no multiplications and no numbers larger than $x^{1/2}$ except the sum. This procedure was developed independently by Keller and Swenson [7] and the present author. It is particularly useful for employing (26). Keller and Swenson present the necessary algorithm and a basic derivation of the process.

For the current paper two methods were used for determining $A_k(x)$. For calculating isolated values, (20) and the above method were used. Special square root routines were used throughout. The time to compute $A_k(x)$ was on the order of

$$T(k, x) \alpha \frac{x^{(k-1)/2}}{k!}.$$

When a large quantity of consecutive values was desired, (8) and (11) were used. An IBM 1301 Disc File was available for additional storage. This Disc File is particularly desirable in allowing the use of one portion of core-storage for computation while data is moved between the Disc File and another portion of core. For the present problem this effectively created a million words of core-storage. The time required to compute $A_k(x)$ for $2 \leqq k \leqq K$ and all integer $x \leqq X$ is

$$T(K, X) \alpha (k - 1) x^{3/2}.$$

Using this method $3\frac{1}{2}$ hours were required for $T(6, 250000)$. Formula (11) was used with the assumption that $R_k(x) < 2^{36}$. This assumption was violated near $R_6(40{,}000)$.

Integer arithmetic was used exclusively for $A_k(x)$ in all programs, and it is expected that all values are correct. Complete agreement was noted for all values published in [7]. Similar agreement existed with [4] except for $A_3(1800^2)$, the largest argument published in that paper. This value was calculated twice for this paper, each calculation requiring 3 minutes.

4. Conclusions. Table 1 gives the first fifty true extreme points for $k = 2, 3$. The number of extreme points for $x \leqq 250{,}000$ ($x^{1/2} \leqq 500$) is

(27)

k	*number of extreme points*	
2	76	
3	80	
4	170	
5	434	
6	474	$(x < 40{,}000)$.

TABLE 1

First 50 extreme points for $k = 2, 3$

X_i	$A_2(X_i)$	$P_2(X_i)$	X_i	$A_3(X_i)$	$P_3(X_i)$
1	5	2	1	7	3
2	9	3	2	19	7
5	21	5	5	57	10
10	37	6	6	81	19
20	69	6	14	251	32
24	69	−6	21	437	34
26	89	7	29	691	37
41	137	8	30	739	51
53	177	10	54	1743	81
130	421	13	90	3695	119
149	481	13	134	6619	122
205	657	13	155	8217	134
234	749	14	174	9771	157
287	885	−17	230	14771	160
340	1085	17	234	15155	161
410	1305	17	251	16831	174
425	1353	18	270	18805	221
480	1489	−19	342	26745	252
586	1861	20	374	30551	254
840	2617	−22	461	41755	294
850	2693	23	494	46297	305
986	3125	27	550	54339	309
1680	5249	−29	666	72359	364
1843	5761	−29	750	86407	371
2260	7129	29	810	96969	405
2591	8109	−31	990	131059	580
3023	9465	−32	1890	344859	682
3024	9465	−35	2070	395231	734
3400	10717	36	2486	519963	756
3959	12401	−37	2757	607141	763
3960	12401	−40	2966	677397	776
5182	16237	−43	3150	741509	959
5183	16237	−46	3566	893019	1028
7920	24833	−48	3630	917217	1105
9796	30725	−50	4554	1288415	1120
11233	35237	−53	4829	1406811	1170
14883	46701	−55	5670	1789599	1205
15119	47441	−57	5750	1827927	1550
15120	47441	−60	8154	3085785	1570
19593	61493	−60	8382	3216051	1576
21600	67797	−61	8774	3444439	1851
21603	67805	−63	8910	3524869	1930
21604	67805	−66	10350	4412643	2028
22177	69605	−66	10710	4645127	2404
28559	89653	−68	15734	8269399	2411
28560	89653	−71	15750	8282167	2565
31679	99449	−74	16302	8721339	2675
31680	99449	−77	17550	9741669	2895
38015	119349	−79	23310	14910309	2905
38016	119349	−82	23894	15474065	2940
38017	119349	−85	24174	15746999	3133

This reflects the increasing values of θ_k and perhaps more "regularity" for the higher values of k.

The problem of showing

$$\operatorname*{Lim\,sup}_{i\to\infty} \frac{|P_k(N_i)|}{N_i^\theta} < +\infty$$

is equivalent to showing

$$\operatorname*{Lim\,sup}_{i\to\infty} (\log |P_k(N_i)| - \theta \log N_i) < +\infty.$$

Graphically this corresponds to finding a straight line with slope θ which majorizes the points $(\log N_i, \log |P_k(N_i)|)$. Figure 1 shows the sequence of extreme points for $x \leqq 250{,}000$. Only a sample of the points for $k = 5, 6$ are shown.

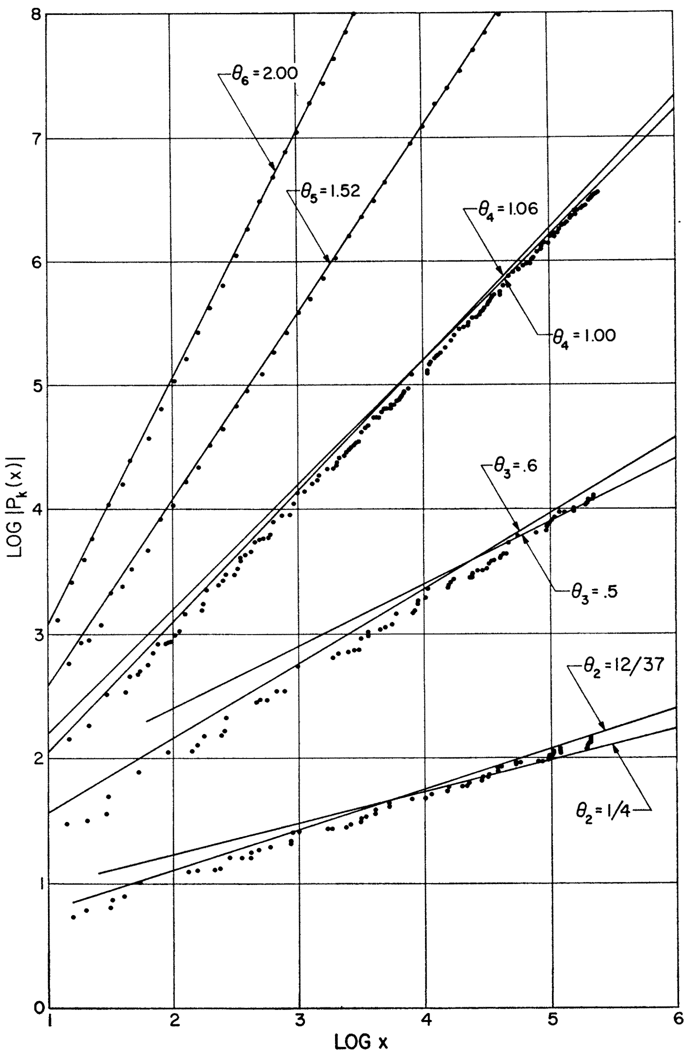

FIGURE 1. Extreme points for $x \leqq 250{,}000$ and $k = 2, 3, 4, 5, 6$.

TABLE 2

$x^{1/2}$	$A_2(x)$	$P_2(x)$	$x^{1/2}$	$A_3(x)$	$P_3(x)$
1000000	3141592649625	−3965	1000	4188781437	−8768
1500000	7068583465945	−4632	1200	7238202017	−27457
2000000	12566370610285	−4074	1400	11494026189	−14133
2500000	19634954076697	−8239	1600	17157266213	−18466
3000000	28274333873841	−8467	1800	24428982249	−42225
3500000	38484509999277	−7198	2000	33510290993	−30645
4000000	50265482451357	−6080	2500	65449818205	−28745
4500000	63617251226505	−8688	3000	113097275709	−59820
5000000	78539816333093	−6652	3500	179594325465	−54565
5500000	95033177762429	−8662	4000	268082474393	−98713
6000000	113097335520185	−9048	4500	381703453381	−54030
6500000	132732289606241	−7928	5000	523598707861	−67737
7000000	153938040012805	−13095	5500	696909887157	−83164
7500000	176714586754401	−10025	6000	904778525345	−158889
8000000	201061929820913	−8834	6500	1150346427953	−82036
8500000	226980069212125	−9738	7000	1436754948853	−91389
9000000	254469004930845	−9928	7500	1767145772565	−95079
9500000	283528736973257	−13222	8000	2144660422929	−161922
9600000	289529178944573	−10262	8500	2572440705977	−78537
9700000	295592452772029	−4235	9000	3053627854381	−204908
9800000	301718558438929	−11835			
9900000	307907495964805	−13531			
10000000	314159265350589	−8390			

The lines drawn represent the minimum slopes which appear to parallel the extreme points. In addition, for $k = 2, 3, 4$ the theoretical minima for θ_k (see (4)) are shown. If θ_k is estimated from these points, the results are

$$\theta_2 \doteqdot .\overline{324} = 12/37,$$
$$\theta_3 \doteqdot .60 = 3/5,$$
$$\theta_4 \doteqdot 1.06, \tag{28}$$
$$\theta_5 \doteqdot 1.52,$$
$$\theta_6 \doteqdot 2.00.$$

The accuracy of visual estimation limits this method to a precision of at most $\pm.01$. For instance in Figure 1, for $k = 2$ a line with slope 12/37 would be indistinguishable from one with slope 1/3. The way which the results for $k = 4, 5, 6$ approach the known values suggests that this method is valuable for the range of x used.

In addition to the true extreme points for $x^{1/2} \leqq 500$ a number of approximate extreme points were calculated from isolated values of $P_2(x)$ and $P_3(x)$. Some of these are shown in Table 2. In Figure 2, the values of $P_2(x)$ for $x^{1/2} \leqq 10{,}000{,}000$ are shown with the true extreme points for $x^{1/2} \leqq 500$. If only the approximate extreme points are considered, one is led to agree with Fraser and Gotlieb [4] that "$\theta_2 = \frac{1}{4}$ is not inconsistent with observed results." But when the distribution of approximate extreme points is considered independently of sampling distribution,

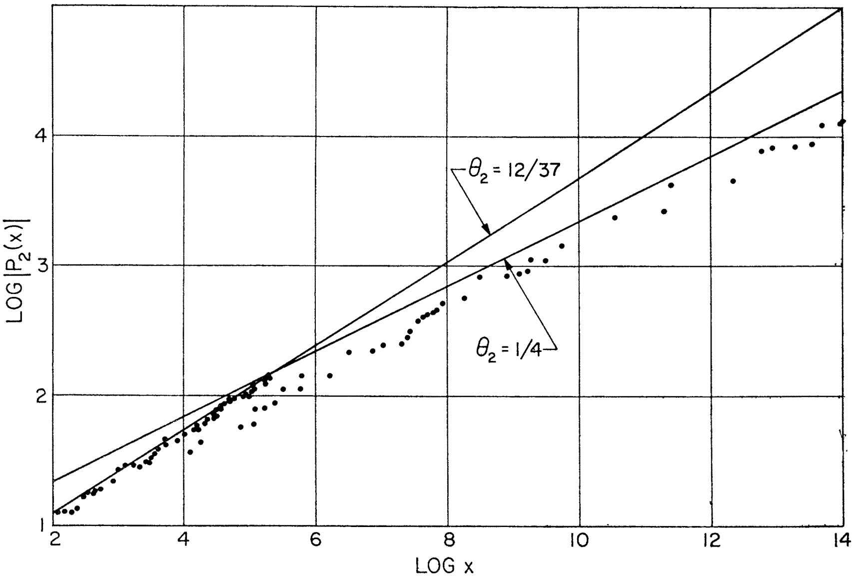

FIGURE 2. Distribution of true and approximate extreme points for $k = 2$.

there is no reason to believe that the true extreme points do not continue near slope 12/37 for $x^{1/2} > 500$. Thus it is not reasonable to conjecture from these results that θ_2 is appreciably less than Chen Jing-ren's bound of 12/37. A logical conjecture based upon these results is $\theta_3 \geqq .3$.

The approximate extreme points for $k = 3$ suggest that $.5 \leqq \theta_3 \leqq .6$, but there are too few points from which to extrapolate with assurance. For instance half of the isolated values qualify as approximate extreme points. The time required to calculate more values of $P_3(x)$ would be prohibitive.

An additional matter of interest is the sign of $P_k(x)$. Keller and Swenson reported that, while most of the values of $P_2(x)$ for integer values of $x^{1/2} \leqq 260{,}000$ were negative, the sign distribution for noninteger $x^{1/2}$ "was about uniform or perhaps even slightly biased in favor of positive values." In this experiment all of the true extreme values for $3400 < x \leqq 250{,}000$ and all of the isolated values for integer $x^{1/2} \leqq 10{,}000{,}000$ were negative.

For $k = 3$, 95% of the true extreme values were positive while the larger isolated values were negative. The four negative extreme values were among the larger extreme points.

For $k = 4, 5, 6$ all of the true extreme values were positive.

Another question is whether or not noninteger values of x would provide different extreme points than the integer values used thus far. From (6) we need only consider $\mathrm{Lim}_{\beta \to 1^-} P_k(n + \beta)$. This question is of little interest for $k = 2$ because $P_2(n + \beta) = P_2(n) - \pi$. However for $k = 3$, for $x < 1000$, the $P_3(n + \beta)$ values were of the same magnitude as the $P_3(n)$ values. For the larger values of k and necessarily smaller x, in accordance with the greater density of extreme points as in (27), there is an alternation of extreme points for small x and a random assortment for larger x.

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