TECHNICAL NOTES AND SHORT PAPERS

# A Note on Equal Sums of Like Powers

By Ian Barrodale

**1. Nature of the Problem.** The numbers 1, 2, and 6 have the same sum and same sum of squares as 0, 4, 5. These two sets are a solution of degree 2 to the problem of sets of integers having equal sums of like powers, i.e., to find integer solutions for the system of equations

$$\text{(A)} \qquad \sum_{i=1}^{s} a_i^j = \sum_{i=1}^{s} b_i^j \qquad (j = 1, 2, \cdots, k).$$

A solution of (A) is written $a_1, \cdots, a_s \overset{k}{=} b_1, \cdots, b_s$ and is said to be of degree $k$. A solution of (A) in which the $a$'s are merely a permutation of the $b$'s is called trivial; we are concerned with nontrivial solutions. Goldbach and Euler noted (1750–1751) that

$$a, b, c, a + b + c \overset{2}{=} a + b, a + c, b + c, 0$$

whereas the following two theorems prove the existence of solutions to (A) for any value of $s$ and $k$.

Theorem 1. *If* $a_1, \cdots, a_s \overset{k}{=} b_1, \cdots, b_s$ *then*

$$Ma_1 + K, \cdots, Ma_s + K \overset{k}{=} Mb_1 + K, \cdots, Mb_s + K$$

*where* $M$, $K$ *are arbitrary integers.*

Theorem 2. *If* $a_1, \cdots, a_s \overset{k}{=} b_1, \cdots, b_s$ *then*

$$a_1, \cdots, a_s, b_1 + d, \cdots, b_s + d \overset{k+1}{=} b_1, \cdots, b_s, a_1 + d, \cdots, a_s + d$$

*for any integer* $d$.

Theorem 1 is due to Frolov [2] and allows one to operate on a solution of (A) according to the rules of elementary algebra, while Theorem 2 is due to Tarry [5] and enables one to build up a solution for (A) of any desired degree.

Thus

$$0, 3 \overset{1}{=} 1, 2$$

gives

$$0, 4, 5 \overset{2}{=} 1, 2, 6 \qquad (d = 3)$$

which gives

$$0, 4, 7, 11 \overset{3}{=} 1, 2, 9, 10 \qquad (d = 5)$$

and so on.

A number of writers have been interested in finding the least value of $s$ for which (A) has solutions for any particular $k$. Bastien [1] proved that $s \geqq k + 1$ and Tarry [5] proved that $s \leqq 2^{k-1}$. Wright [6, p. 261] defined $N(k)$ as the least

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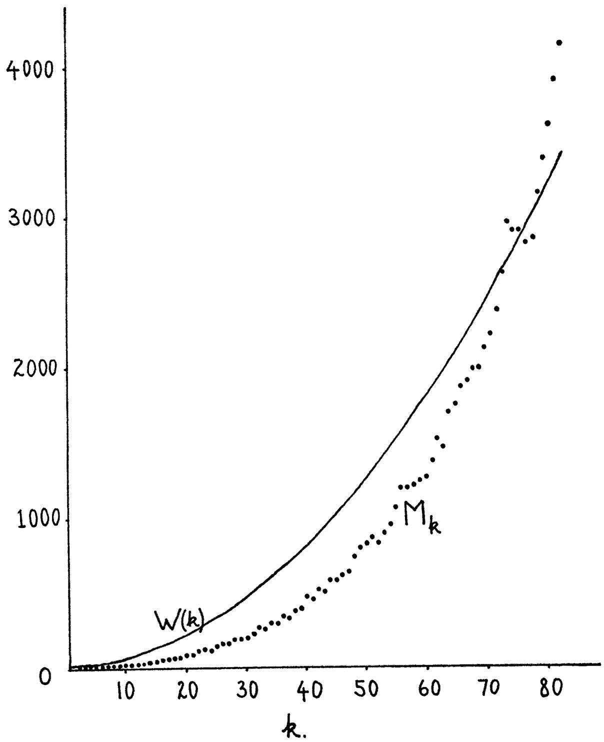


FIGURE 1. Graph of the numerical bounds.

number $N$ such that $a_1, \cdots, a_N \overset{k}{=} b_1, \cdots, b_N$, proved that $N(k) \leqq \frac{1}{2}(k^2+4) = W(k)$, and conjectured that in fact $N(k) = k+1$ for all $k$. This conjecture has been proved by examples for all $k \leqq 9$ but for no other degree [3, p. 338].

**2. Results.** In an extension of the problem of equal sums of like powers Wright [6, p. 261] has defined $M(k)$ as the smallest number $M$ such that (A) has a solution with

$$a_1^{k+1} + \cdots + a_M^{k+1} \neq b_1^{k+1} + \cdots + b_M^{k+1}$$

and proved that $M(k) \leqq N(k^2)$. Clearly $M(k) \geqq N(k) \geqq k+1$ and in particular $M(k) = N(k) = k+1$ for all $k \leqq 9$. We now prove Theorem 3 which is a modification of a result due to Melzak [4, p. 234].

THEOREM 3.

$$M(k) = \tfrac{1}{2} \min_{P \in \Omega'} S[P(x)(1-x)^{k+1}]$$

*where $\Omega'$ is the class of all polynomials whose coefficients are integers, not all zero, and furthermore if $P \in \Omega'$ then $P(1) \neq 0$; and*

$$S[P] = \sum_{i=0}^{n} |a_i| \quad for \quad P = P(x) = \sum_{i=0}^{n} a_i x^i.$$

*Proof.* For every $P \in \Omega'$, $P(x)(1-x)^{k+1}$ is the generating function for a solution

TABLE I

*Numerical Bounds for the Extended Problem*

| $k$ | $M_k$ | $p$ | $k$ | $M_k$ | $p$ |
|---|---|---|---|---|---|
| 2 | 3 | 1 | 44 | 588 | 4 |
| 3 | 4 | 1 | 45 | 588 | 3 |
| 4 | 6 | 1 | 46 | 627 | 4 |
| 5 | 6 | 1 | 47 | 644 | 4 |
| 6 | 10 | 2 | 48 | 742 | 4 |
| 7 | 12 | 1 | 49 | 802 | 4 |
| 8 | 18 | 1 | 50 | 830 | 4 |
| 9 | 18 | 1 | 51 | 872 | 4 |
| 10 | 22 | 1 | 52 | 834 | 4 |
| 11 | 22 | 1 | 53 | 896 | 5 |
| 12 | 30 | 2 | 54 | 958 | 5 |
| 13 | 32 | 1 | 55 | 1072 | 5 |
| 14 | 41 | 1 | 56 | 1202 | 5 |
| 15 | 46 | 1 | 57 | 1206 | 4 |
| 16 | 58 | 1 | 58 | 1218 | 4 |
| 17 | 58 | 1 | 59 | 1248 | 5 |
| 18 | 68 | 2 | 60 | 1270 | 5 |
| 19 | 74 | 1 | 61 | 1376 | 5 |
| 20 | 88 | 2 | 62 | 1517 | 5 |
| 21 | 92 | 2 | 63 | 1464 | 5 |
| 22 | 119 | 2 | 64 | 1694 | 5 |
| 23 | 124 | 2 | 65 | 1750 | 5 |
| 24 | 118 | 2 | 66 | 1866 | 5 |
| 25 | 146 | 2 | 67 | 1902 | 5 |
| 26 | 159 | 2 | 68 | 1990 | 5 |
| 27 | 166 | 3 | 69 | 1994 | 5 |
| 28 | 196 | 2 | 70 | 2120 | 6 |
| 29 | 198 | 3 | 71 | 2224 | 6 |
| 30 | 207 | 2 | 72 | 2372 | 6 |
| 31 | 228 | 3 | 73 | 2618 | 6 |
| 32 | 274 | 2 | 74 | 2947 | 6 |
| 33 | 258 | 3 | 75 | 2906 | 6 |
| 34 | 305 | 3 | 76 | 2902 | 6 |
| 35 | 308 | 3 | 77 | 2822 | 6 |
| 36 | 344 | 3 | 78 | 2853 | 6 |
| 37 | 332 | 3 | 79 | 3150 | 7 |
| 38 | 381 | 3 | 80 | 3386 | 6 |
| 39 | 402 | 3 | 81 | 3604 | 7 |
| 40 | 472 | 3 | 82 | 3903 | 7 |
| 41 | 462 | 3 | 83 | 4136 | 7 |
| 42 | 525 | 4 | 84 | 4502 | 7 |
| 43 | 514 | 3 | 85 | 4547 | 7 |

of degree $k$ to the extended problem. For suppose $P(1) = 0$, then $P(x)(1 - x)^{k+1} = Q(x)(1 - x)^{k+2}$ which generates a solution to (A) of degree $k + 1$. Q.E.D.

Unfortunately Theorem 3 has not allowed us to prove $M(k) = N(k) = k + 1$ for any $k \geqq 10$ but for every $P \in \Omega'$ it does provide a solution to the extended problem and an estimate $M_k$ for $M(k)$. Using an IBM 1620 we evaluated the expression $\frac{1}{2}S[P(x)(1 - x)^{k+1}]$ for various $P \in \Omega'$ and $1 \leqq k \leqq 40$, obtaining the lowest estimates $M_k$ when $P(x)$ was of the form

TABLE II

*Smaller Solutions for Certain Degrees*

| $k$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $s$ | 14 | 18 | 24 | 30 | 30 | 30 | 38 | 48 | 58 | 58 | 65 | 80 | 84 |

TABLE III

*Example Illustrating a Weakness in the Algorithm*

| $k$ | $d$ | $s$ | $\frac{1}{2}S\left[(1-x)\prod_{j=1}^{k+1}(1-x^j)\right]$ |
|---|---|---|---|
| 1 | 2 | 2 | 2 |
| 2 | 3 | 3 | 3 |
| 3 | 5 | 4 | 4 |
| 4 | 4 | 6 | 6 |
| 5 | 7 | 6 | 6 |
| 6 | 11 | 8 | 11 |
| 7 | 9 | 10 | 12 |
| 8 | 13 | 14 | 18 |
| 9 | 17 | 14 | 18 |
| 10 | 19 | 18 | 22 |
| 11 | | 24 | 22 |

$$P(x) = \left[\prod_{j=1}^{p}(1-x^j)\right]\left[\prod_{n=1}^{k}\sum_{j=0}^{n}x^j\right]$$

where $1 \leqq p \leqq 7$. Then

$$M_k = \tfrac{1}{2}S\left[Q(x)\prod_{j=1}^{k+1}(1-x^j)\right]$$

where

$$Q(x) = \prod_{j=1}^{p}(1-x^j).$$

Table I was formed by selecting the lowest estimate $M_k$ for $2 \leqq k \leqq 85$ and inserting the value of $p$ relevant to each $k$. Figure 1 is the graph of Table I together with the graph of the best upper bound for $N(k)$ (since $M(k) = N(k)$ for $k \leqq 9$ it is perhaps more realistic to compare the estimates $M_k$ to bounds for $N(k)$). It is obvious from Figure 1 that while the estimates $M_k$ are lower than $W(k)$ for $2 \leqq k \leqq 73$ they soon become larger than $W(k)$. Hence if this method is to give further useful results new multipliers $P(x)$ are needed.

Using Theorem 2 we attempted to obtain solutions to (A) for $k \geqq 10$ where the number of terms $s$ is less than the estimates $M_k$ given in Table I. We programmed a computer so that it would read a solution to (A) of any reasonable length and degree, and then calculate the difference $d$ that occurs most frequently between any two terms from the same side of this given solution. It would then use $d$ with Theorem 2 to produce a solution to (A) of the next higher degree, and continue in this manner. By considering solutions to (A) of different lengths and degrees we have found examples of solutions for $10 \leqq k \leqq 22$ where the number of terms $s$ is less than those given in Table I. Table II gives the value of $s$ corresponding to each value of $k$.

However this method of producing solutions to (A) with a small number of terms is subject to the following weakness. We had assumed that from any particular solution to (A) solutions of higher degree would be generated containing the least number of terms $s$, so long as the most frequent difference $d$ was used at each step. After producing the following results this assumption was seen to be false.

When forming Table I the multiplier $(1 - x)$ was used with $\prod_{j=1}^{11} (1 - x^j)$ to produce a solution to the extended problem where $s = 22$ for $k = 11$. This is equivalent to starting with the solution $0, 2 \stackrel{1}{=} 1, 1$ and using Theorem 2 with $d = 2, 3, \cdots, 11$. Table III compares the lengths of the solutions generated in this manner with those generated from the same initial solution but using the most frequent difference $d$ at each step.

Thus, by a more careful choice of $d$, the length of solutions can be decreased for $k = 6, 7, 8, 9, 10$. But for $k = 11$ this produces a solution to the extended problem where $s = 24$. This solution is longer than that obtained from a sequence of solutions which was constructed from values for $d$ that did not always represent the most frequent difference.

Finally, although solutions to (A) for $k = 6$ and $s = 7$ exist, we proved that no such solution can be obtained from a sequence generated by any solution for $k = 1$ and $s = 2$ using the most frequent difference $d$ at each step.

Although Theorem 2 was used to generate most solutions for $k \leqq 9$ where $s = k + 1$, it appears that for $k \geqq 10$ it alone will not be sufficient.

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# Numerical Solutions of the Diophantine Equation $y^3 - x^2 = k$

By M. Lal, M. F. Jones and W. J. Blundon

**Introduction.** The distribution of squares and cubes differing by a given integer is very interesting [1] and has attracted many mathematicians over the past few centuries. Probably this is due to the fact that $y^3 - x^2 = k$ is the simplest of all nontrivial Diophantine equations of degree greater than two. The solution of this equation is equivalent to the problem of representation of numbers by binary cubic

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