pression is $4n^2/(\log_2 n - \log_2 \log_2 n - 1)$. The result stated at the beginning of this section now follows immediately.

4. Generalizations. The corresponding problem may be considered for n by n matrices whose entries are taken from the integers $0, 1, \cdots, k-1$. If an operation on such a matrix consists of adding a multiple of some row to some other row modulo k, then it can be shown that the foregoing theorem remains valid in this more general situation for any fixed value of k. In fact, the bounds in Sections 2 and 3 will still hold if $\log_2 n$ is replaced by $\log_k n$.

1. N. J. Fine & I. Niven, "The probability that a determinant be congruent to a (mod m)," *Bull. Amer. Math. Soc.*, v. 50, 1944, pp. 89–93.

Evaluation of $I_n(b) = 2\pi^{-1} \int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos(bx)\, dx$ and of Similar Integrals

By Rory Thompson

Medhurst and Roberts [1] suggest the problem of evaluating $I_n(b)$ for non-integral values of b. There will be developed in this note an effective recursion scheme for such a calculation. In particular, it can be used to evaluate $I_n(0)$ for moderate values of n.

Following a suggestion by Hamming [2, p. 164], we differentiate $I_n(b)$ with respect to the parameter b, which is permissible by virtue of uniform convergence of the resulting integral for $n > 2$ and continuity of the corresponding integrand with respect to both x and b.

Thus we obtain

$$\begin{aligned} I_n'(b) &= -2\pi^{-1} \int_0^\infty \left(\frac{\sin x}{x}\right)^{n-1} \sin x \sin(bx)\, dx \\ &= \pi^{-1} \int_0^\infty \left(\frac{\sin x}{x}\right)^{n-1} [\cos(b+1)x - \cos(b-1)x]\, dx \\ &= \frac{1}{2}\left[I_{n-1}(b+1) - I_{n-1}(b-1)\right]. \end{aligned}$$

If the first expression for $I_n'(b)$ is integrated by parts there results the relation

$$\begin{aligned} I_n'(b) &= (n-1)b^{-1}\, 2\pi^{-1} \int_0^\infty \left(\frac{\sin x}{x}\right)^{n} \cos(bx)\, dx \\ &\quad - nb^{-1}\, 2\pi^{-1} \int_0^\infty \left(\frac{\sin x}{x}\right)^{n-1} \cos x \cos(bx)\, dx \\ &= \frac{n-1}{b} I_n(b) - \frac{n}{2b}\left[I_{n-1}(b+1) + I_{n-1}(b-1)\right]. \end{aligned}$$

Received June 14, 1965. Revised August 19, 1965.

Elimination of $I_n'(b)$ between these expressions yields the recurrence relation

$$I_n(b) = \frac{1}{2(n-1)}\Big[(n+b)I_{n-1}(b+1) + (n-b)I_{n-1}(b-1)\Big].$$

This recursion scheme for calculating $I_n(b)$ is reasonably stable, owing to the addition of comparable positive numbers and to the fact that an error in either $I_{n-1}(b+1)$ or $I_{n-1}(b-1)$ is multiplied by at most $(n+b)/2(n-1)$, which is less than unity, since $b < n$ for nonzero values of $I_n(b)$.

Starting values are readily given by the relations

$$\begin{aligned} I_3(b) &= \tfrac{1}{8}[(b+3)^2 - 3(b+1)^2], \qquad 0 \leqq b \leqq 1 \\ &= \tfrac{1}{8}[(b+3)^2 - 3(b+1)^2 + 3(b-1)^2], \qquad 1 \leqq b \leqq 3 \\ &= 0, \qquad b \geqq 3. \end{aligned}$$

Furthermore, we observe that negative values of b can be taken into account by use of the relation

$$I_n(-b) = I_n(b).$$

The iterative procedure just described was used to produce an 8D table of $I_n(b)$ for $n = 3(1)100$, $b = 0(0.1)9$ in approximately 0.9 minute on an IBM 7094 system at the MIT Computation Center. This table has been deposited in the UMT file of this journal.

For integral values of b the table was checked by using the recurrence formula in the form

$$I_{n-1}(b+1) = \frac{2(n-1)}{n+b} I_n(b) - \frac{n-b}{n+b} I_{n-1}(b-1).$$

This implies

$$I_{n-1}(1) = \frac{n-1}{n} I_n(0).$$

Values of $I_n(0)$ may be obtained from 10D tables in [1] and [3], so that $I_n(b)$ can be evaluated for integral values of b in a more compact form than the formulas in [1].

The use of recurrence formulas is applicable to the numerical evaluation of other integrals, including indefinite ones. One such example is the calculation of the chi-square distribution, which was accomplished by Harter [4] essentially by direct integration. In this case, integrating

$$F_n(u) = \left[2^{n/2}\Gamma\left(\frac{n}{2}\right)\right]^{-1} \int_0^u \exp\left(-\frac{x}{2}\right) x^{n/2-1}\, dx$$

by parts yields the recurrence relation

$$F_n(u) = F_{n-2}(u) - H_n(u),$$

where

$$H_n(u) = u^{n/2-1} \Big/ \left[2^{n/2-1} \exp\left(\frac{u}{2}\right) \Gamma\left(\frac{n}{2}\right)\right] = \frac{u}{n-2} H_{n-1}(u)$$

and

$$H_1(u) = \exp\left(-\frac{u}{2}\right),$$

$$F_2(u) = 1 - \exp\left(-\frac{u}{2}\right).$$

This procedure for evaluating $F_n(u)$ is sufficiently fast to permit a direct search for percentage points, in lieu of interpolation. Thus eleven critical levels were calculated to $5D$ for $n = 2(2)100$ in 1.8 minutes on an IBM 7094.

Many other types of integrals exist for which this recursion scheme is feasible, in particular, Fourier (and other) transforms similar to $I_n(b)$.

Massachusetts Institute of Technology
Cambridge, Massachusetts

1. R. G. Medhurst & J. H. Roberts, "Evaluation of the integral $I_n(b) = 2/\pi \int_0^\infty ((\sin x)/x)^n \cos(bx)\, dx$," *Math. Comp.*, v. 19, 1965, pp. 113–117.

2. R. W. Hamming, *Numerical Methods for Scientists and Engineers*, International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1962. MR **25** #735.

3. K. Harumi, S. Katsura & J. W. Wrench, Jr., "Values of $2/\pi \int_0^\infty ((\sin t)/t)^n\, dt$," *Math. Comp.*, v. 14, 1960, p. 379. MR **22** #12737.

4. H. L. Harter, *New Tables of the Incomplete Gamma-Function Ratio and of Percentage Points of the Chi-Square and Beta Distributions*, U. S. Government Printing Office, Washington, D. C., 1964. MR **30** #1562.

Evaluation of Some Integrals Involving the ψ-Function

By M. L. Glasser

In the Bateman manuscript project tables, Erdelyi et al. [1] list five integrals over the unit interval involving the ψ-function (logarithmic derivative of the gamma function). The first of these is trivial, the second is easily derived by integrating by parts to derive a differential equation in terms of the parameter a. The fourth and fifth formulas are obtained by equating the imaginary and real parts of the second and the third is simply the case $a = 0$ of the fourth. The purpose of this note is to point out that this table can be easily extended by simple use of the properties of the ψ-function. For example, many convergent integrals of the form

$$I = \int_m^n f(x)\psi(x)\, dx,$$

where m and n are integers and $f(x)$ is a function such that $f(x) = -f(m + n - x)$, can be evaluated exactly. Thus, by symmetry

$$I = \tfrac{1}{2} \int_m^n f(x)\{\psi(x) - \psi(m + n - x)\}\, dx.$$

Now use of the relations $\psi(y + 1) = \psi(y) + y^{-1}$ and $\psi(y) - \psi(1 - y) = -\pi \cot \pi y$ gives immediately

Received August 26, 1965.