and

$$H_1(u) = \exp\left(-\frac{u}{2}\right),$$

$$F_2(u) = 1 - \exp\left(-\frac{u}{2}\right).$$

This procedure for evaluating $F_n(u)$ is sufficiently fast to permit a direct search for percentage points, in lieu of interpolation. Thus eleven critical levels were calculated to $5D$ for $n = 2(2)100$ in 1.8 minutes on an IBM 7094.

Many other types of integrals exist for which this recursion scheme is feasible, in particular, Fourier (and other) transforms similar to $I_n(b)$.

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1. R. G. Medhurst & J. H. Roberts, "Evaluation of the integral $I_n(b) = 2/\pi \int_0^\infty ((\sin x)/x)^n \cos (bx)\, dx$," *Math. Comp.*, v. 19, 1965, pp. 113–117.
2. R. W. Hamming, *Numerical Methods for Scientists and Engineers*, International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1962. MR **25** #735.
3. K. Harumi, S. Katsura & J. W. Wrench, Jr., "Values of $2/\pi \int_0^\infty ((\sin t)/t)^n\, dt$," *Math. Comp.*, v. 14, 1960, p. 379. MR **22** #12737.
4. H. L. Harter, *New Tables of the Incomplete Gamma-Function Ratio and of Percentage Points of the Chi-Square and Beta Distributions*, U. S. Government Printing Office, Washington, D. C., 1964. MR **30** #1562.

Evaluation of Some Integrals Involving the ψ-Function

By M. L. Glasser

In the Bateman manuscript project tables, Erdelyi et al. [1] list five integrals over the unit interval involving the ψ-function (logarithmic derivative of the gamma function). The first of these is trivial, the second is easily derived by integrating by parts to derive a differential equation in terms of the parameter a. The fourth and fifth formulas are obtained by equating the imaginary and real parts of the second and the third is simply the case $a = 0$ of the fourth. The purpose of this note is to point out that this table can be easily extended by simple use of the properties of the ψ-function. For example, many convergent integrals of the form

$$I = \int_m^n f(x)\psi(x)\, dx,$$

where m and n are integers and $f(x)$ is a function such that $f(x) = -f(m + n - x)$, can be evaluated exactly. Thus, by symmetry

$$I = \tfrac{1}{2} \int_m^n f(x)\{\psi(x) - \psi(m + n - x)\}\, dx.$$

Now use of the relations $\psi(y + 1) = \psi(y) + y^{-1}$ and $\psi(y) - \psi(1 - y) = -\pi \cot \pi y$ gives immediately

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$$I = \int_m^n f(x)R(x)\,dx - \frac{\pi}{2}⨍_m^n f(x)\cot \pi x\,dx$$

where $R(x)$ is rational and the slash denotes the Cauchy principal part. When $f(x)$ is rational or trigonometric these integrals can frequently be expressed in familiar terms. As an example we consider the case $f(x) = x(1 - x)\cos \pi x$, $m = 0$, $n = 1$. Proceeding as above and noting that

$$\int_0^{\pi/2} x \csc x\,dx = 2\beta(2), \qquad \int_0^{\pi/2} x^2 \csc x\,dx = 2\pi\beta(2) - \frac{7}{2}\zeta(3),$$

where $\beta(2)$ is Catalan's constant and ζ represents the Riemann zeta function, we find

$$\int_0^1 f(x)\psi(x)\,dx = \frac{2}{\pi^2} - \frac{7}{2\pi^2}\zeta(3).$$

Now by making use of the properties of the ψ-function we also obtain, e.g.,

$$\int_0^1 f(x)\psi(-x)\,dx = \frac{7}{2\pi^2}\zeta(3),$$

$$\int_0^1 f(x)\psi\left(x + \frac{1}{2}\right)dx = \frac{6}{\pi^2} - \frac{1}{2}Si\frac{\pi}{2},$$

$$\int_0^1 f(x)\psi\left(x - \frac{1}{2}\right)dx = \frac{4}{\pi^2},$$

$$\int_0^1 f(x)\left\{\psi\left[\frac{x+1}{2}\right] + \psi\left(\frac{x}{2}\right)\right\}du = \frac{4}{\pi^2} - \frac{7}{\pi^2}\zeta(3).$$

It is interesting to note that $\int_0^1 x(1 - x)\cos \pi x\psi(x/2)\,dx$ appears to be inexpressible in similar closed form.

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1. Erdelyi et al., *Tables of Integral Transforms*, Vol. II, McGraw-Hill, New York, 1954, p. 305. MR **16**, 468.