A Linear Three-Level Difference Scheme for Quasilinear Parabolic Equations*

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The purpose of this paper is to establish a convergence theorem for a second order accurate, linear, three-level difference scheme for a class of quasilinear parabolic differential equations. In the interests of notational simplicity, we shall present our result in detail only for the special semilinear partial differential equation

$$(1a) \qquad b(w)\,\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(a(w)\,\frac{\partial w}{\partial x}\right).$$

Later, we shall indicate how our convergence theorem can be extended to more general equations than (1a).

Let $w = w(x, t)$ be a *smooth* solution of the differential equation in the rectangular region $R = \{(x, t) : 0 \leq x \leq 1, 0 \leq t \leq T\}$ such that

$$
\begin{aligned}
w(x, 0) &= \phi(x), & 0 &\leq x \leq 1,\\
(1b) \qquad w(0, t) &= \psi_0(t), & 0 &\leq t \leq T,\\
w(1, t) &= \psi_1(t), & 0 &\leq t \leq T,
\end{aligned}
$$

where ϕ, ψ_0 and ψ_1 are specified functions. In addition to assuming that the coefficient functions b and a are defined and *smooth* for all real w, we assume that there exist positive constants μ and η such that

$$(2) \qquad b(w) \geq \mu > 0 \quad \text{and} \quad a(w) \geq \eta > 0.$$

Because of (2), it follows that the differential equation (1a) is uniformly parabolic, so that w is the only solution of the boundary value problem (1).

Before describing the three-level difference scheme for (1), we introduce some convenient notation. If N is a positive integer and $h > 0$ is defined by the relation $(N + 1)h = 1$, then we put $X = \{sh : s = 0, 1, \cdots, N + 1\}$. If $0 < 2k < T$, then we let J be the largest positive integer such that $kJ \leq T$. We assume that there are two positive constants λ_- and λ_+ such that all admissible h and k satisfy

$$(3) \qquad \lambda_- h \leq k \leq \lambda_+ h.$$

The standard difference operators D_+, D_- and D_0 are defined, as follows:

$$
\begin{aligned}
hD_+u(x) &= u(x + h) - u(x),\\
hD_-u(x) &= u(x) - u(x - h),\\
2hD_0u(x) &= u(x + h) - u(x - h).
\end{aligned}
$$

Finally, for any suitably defined function $v(x)$, we put

$$\bar{a}(v(x)) = a\{[v(x) + v(x - h)]/2\}.$$

We determine a sequence u_0, u_1, $\cdots$, u_j of real valued functions defined on the $(N + 2)$-element set X, as follows: For each j we put $u_j(0) = \psi_0(jk)$ and $u_j(1) = \psi_1(jk)$. Then, on the N-element set $X \cap (0, 1)$,

$$(4a) \qquad u_0 = \phi,$$

$$(4b) \qquad u_1 = \phi + \frac{k}{b(\phi)} (a(\phi)\phi')',$$

and, for $1 \leqq j < J$,

$$(4c) \qquad b(u_j)(u_{j+1} - u_{j-1}) = 2kD_+(\bar{a}(u_j)D_-\hat{u}_j),$$

where

$$(5) \qquad \hat{u}_j = (u_{j+1} + u_j + u_{j-1})/3.$$

It is clear that the three-level difference equation (4c) determines u_{j+1} uniquely as the solution of a linear, numerically stable, tridiagonal system of equations. The relations (4a) and (4b) provide the necessary starting values for (4c). Although we give no details, it will become obvious that it is permissible to replace (4b) by

$$u_1 = \phi + \frac{k}{b(\phi)} [D_+(\bar{a}(\phi)D_-\phi)].$$

For the approximation determined by (5) we have the convergence theorem.

THEOREM. *There is a constant $A > 0$, independent of h, k and u_j such that*

$$(6) \qquad \max_{x \in X; 1 \leqq j \leqq J} | u_j(x) - w(x, jk)| \leqq A(h^2 + k^2)$$

for all sufficiently small h and k satisfying (3).

Previously, the weaker error estimate

$$(7) \qquad \max_{1 \leqq j \leqq J} \left(h \sum_{s=1}^{N} | u_j(sh) - w(sh, jk)|^2 \right)^{1/2} \leqq A(h^2 + k^2)$$

was proved for certain mild (linear) generalizations of the heat equation [1], [3]. Also, in a private communication, James Gunn showed us a derivation of the error estimate (7) for equations of the form

$$b(x, t) \frac{\partial w}{\partial t} = \frac{\partial}{\partial x} \left(a(x, t, w) \frac{\partial w}{\partial x} \right) + c(x, t, w).$$

His proof, however, assumes, in addition to the usual parabolicity inequalities, that $\partial c/\partial w$ is bounded for $(x, t) \in R$ and $| w | < +\infty$ and a has continuous second and bounded third derivatives in the same region.

Let us denote by Ω_h the real linear space of all real-valued functions $u(x)$ defined on the discrete set X for which $u(0) = u(1) = 0$. It is obvious that Ω_h has dimension N. The l_2-inner product (u, v) and induced norm $\| u \| = (u, v)^{1/2}$ on Ω_h are defined by

$$(u, v) = h \sum_{s=1}^{N} u(sh)v(sh).$$

The l_∞-norm $\| u \|_\infty$ on Ω_h is given by

$$\| u \|_\infty = \max_{1 \leqq s \leqq N} | u(sh)|.$$

The D-inner product $(u, v)_D$ and induced D-norm $\| u \|_D = (u, v)_D^{1/2}$ on Ω_h are defined by

$$(u, v)_D = h \sum_{s=1}^{N+1} D_-u(sh)D_-v(sh).$$

It is well known [2] that the foregoing norms satisfy the inequalities

(8) $$\| u \| \leqq \| u \|_\infty \leqq \| u \|_D .$$

For the solution w of (1) we put $w_j(x) = w(x, jk)$. We define on Ω_h a modified D-inner product $(u, v)_{j,D}$ and induced norm $\| u \|_{j,D} = (u, v)_{j,D}^{1/2}$ by

$$(u, v)_{j,D} = h \sum_{s=1}^{N+1} \bar{a}(u_j(sh))D_-u(sh)D_-v(sh).$$

It follows immediately from (2) that

(9) $$\| u \|_{j,D} \geqq \eta^{1/2} \| u \|_D , \qquad\qquad 0 \leqq j \leqq J$$

Furthermore, if

$$A_1 = \sup_R \left| \frac{\partial w}{\partial t} \right| \sup_R | a'(w(x, t)) |,$$

then from the mean value theorem and (9) we obtain the inequalities

(10) $$|(u, v)_{j,D} - (u, v)_{j-1,D} | \leqq kA_1 \| u \|_D \| v \|_D ,$$

(11) $$\| u \|_{j,D}^2 \leqq (1 + k\eta^{-1}A_1) \| u \|_{j-1,D}^2 .$$

Finally, summation by parts shows that [2]

(12) $$(u, D_+(\bar{a}(w_j)D_-v)) = - (u, v)_{j,D}$$

for any $u, v \in \Omega_h$.

With the aid of these preliminary results, we proceed to the proof of the convergence theorem. The usual argument, based on Taylor's formula, shows that the solution w of (1) satisfies, in $X \cap (0, 1)$, the relations:

(13a) $$w_0 = \phi,$$

(13b) $$w_1 = u_1 + \tau_1 ,$$

and, for $1 \leqq j < J$,

(13c) $$b(w_j)(w_{j+1} - w_{j-1}) = 2kD_+(\bar{a}(w_j)D_-\hat{w}_j) + 2k\tau_{j+1} ,$$

where $\hat{w}_j$ is defined as in (5) and the local truncation errors τ_j are such that

(14) $$\| \tau_j \|_\infty \leqq A_2(h^2 + k^2), \qquad\qquad 1 \leqq j \leqq J,$$

for a suitable constant A_2 . Without loss of generality, we may assume also that

(15) $$\| w_1 - u_1 \|_D \leqq A_2(h^2 + k^2).$$

Obviously, the error function $z_j = w_j - u_j$ belongs to the space Ω_h , and it

follows from (4), (13) and the mean value theorem that

(16a)
$$z_0 = 0,$$

(16b)
$$z_1 = \tau_1,$$

and, for $1 \leqq j < J$,

(16c)
$$
\begin{aligned}
b(u_j)(z_{j+1} - z_{j-1}) &= 2kD_+(\bar{a}(w_j)D_-\hat{z}_j) + 2k\tau_{j+1} \\
&\quad + b'(\Theta_j)z_j(w_{j+1} - w_{j-1}) \\
&\quad + 2kD_+(\sigma_j D_-\hat{w}_j) - 2kD_+(\sigma_j D_-\hat{z}_j),
\end{aligned}
$$

where

(17)
$$\sigma_j(x) = \bar{a}(w_j(x)) - \bar{a}(u_j(x)), \qquad x \in X - \{0\},$$

and

(18)
$$|\Theta_j| \leqq \sup_R |w| + \|z_j\|_\infty.$$

First, we choose h and k so small that

$$A_2(1 + \lambda_-^{-2})k^2 \leqq k \leqq 1.$$

Then, using (3), (8) and (15) we find that

(19)
$$\|z_1\|_\infty \leqq \|z_1\|_D \leqq k \leqq 1.$$

Now, we prove by induction that $\|z_j\|_\infty \leqq k$ for all sufficiently small h and k satisfying (3). It follows from (16a) and (19) that the assertion is true for $j = 0$ and $j = 1$. Assume that $\|z_l\|_\infty \leqq k$ for all nonnegative integers $l \leqq j$, and we shall prove that $\|z_{j+1}\|_\infty \leqq k$.

Putting $z_j^* = z_{j+1} - z_{j-1}$, we form the l_2-inner product of (16c) with $3z_j^*$ to obtain

(20)
$$I_1 + I_2 = \sum_{r=3}^{6} I_r,$$

where

$$
\begin{aligned}
I_1 &= 3(z_j^*, b(u_j)z_j^*), \\
I_2 &= -6k(z_j^*, D_+(\bar{a}(w_j)D_-\hat{z}_j)), \\
I_3 &= 6k(z_j^*, \tau_{j+1}), \\
I_4 &= 3(z_j^*, b'(\Theta_j)z_j\{w_{j+1} - w_{j-1}\}), \\
I_5 &= 6k(z_j^*, D_+(\sigma_j D_-\hat{w}_j)), \\
I_6 &= -6k(z_j^*, D_+(\sigma_j D_-\hat{z}_j)).
\end{aligned}
$$

It follows immediately from (2) that

(21a)
$$I_1 \geqq 3\mu \|z_j^*\|^2.$$

Using (5) and (12) we obtain the relation

$$I_2 = 2k[\| z_{j+1} \|_{j,D}^2 + (z_{j+1}, z_j)_{j,D} - \| z_{j-1} \|_{j,D}^2 - (z_j, z_{j-1})_{j,D}].$$

If we put

(22) $$L_j = 2[\| z_{j+1} \|_{j,D}^2 + (z_{j+1}, z_j)_{j,D} + \| z_j \|_{j,D}^2],$$

then

$$I_2 = kL_j - 2k[\| z_j \|_{j,D}^2 + (z_j, z_{j-1})_{j,D} + \| z_{j-1} \|_{j,D}^2].$$

Using (10), (11) and Schwarz's inequality we obtain the estimate

(21b) $$I_2 \geqq kL_j - k(1 + kA_3)L_{j-1},$$

where

$$A_3 = 3\eta^{-1}A_1.$$

Using the generalized Cauchy-Schwarz inequality on I_3 we find that

(21c) $$| I_3 | \leqq \epsilon \| z_j^* \|^2 + 9k^2\epsilon^{-1} \| \tau_{j+1} \|^2$$

for any positive number ϵ.

Our induction assumption $\| z_j \|_\infty \leqq k \leqq 1$ and (18) imply that $| \Theta_j | \leqq m + 1$, where $m = \sup_R | w |$; hence, if

$$A_4 = \tfrac{3}{2} \sup_{|w| \leqq m+1} | b'(w)| \sup_R \left| \frac{\partial w}{\partial t} \right|,$$

then

$$| I_4 | \leqq 2kA_4 \| z_j^* \| \| z_j \|.$$

Therefore, by (8)

(21d) $$| I_4 | \leqq \epsilon \| z_j^* \|^2 + k^2A_4^2\epsilon^{-1} \| z_j \|_D^2.$$

Since our induction assumption implies that $\| u_j \|_\infty \leqq \| w_j \|_\infty + \| z_j \|_\infty \leqq m + 1$, using (17) we can find a constant $A_5 > 0$, depending only on λ_+, $\sup_R | \partial w/\partial x |$, $\sup_R | \partial^2 w/\partial x^2 |$, and

$$\sup_{|w| \leqq m+1} [| a'(w) |, | a''(w) |],$$

such that

$$\| D_+(\sigma_j D_- \hat{w}_j) \| \leqq A_5 \| z_j \|_D ;$$

hence, for any $\epsilon > 0$, we have the estimate

(21e) $$| I_5 | \leqq \epsilon \| z_j^* \|^2 + 9k^2\epsilon^{-1}A_5^2 \| z_j \|_D^2.$$

Finally, if

$$A_6 = 3\eta^{-1} \sup_{|w| \leqq m+1} | a'(w) |,$$

then, using summation by parts and Schwarz's inequality, we find that

(21f) $$| I_6 | \leqq k^2A_6(L_j + L_{j-1}).$$

Now, from (20) and (21) we obtain the estimate

$$(22) \quad (3\mu - 3\epsilon)\| z_j^* \|^2 + k(1 - kA_6)L_j \leqq k(1 + kA_7)L_{j-1} \\ + 9k^2\epsilon^{-1}\| \tau_{j+1} \|^2 + k^2 A_8 \epsilon^{-1}\| z_j \|_D^2,$$

where

$$A_7 = A_3 + A_6 \quad \text{and} \quad A_8 = A_4^2 + 9A_5^2.$$

Choose $\epsilon = \mu$. Then (22) implies that

$$(23) \quad (1 - kA_6)L_j \leqq (1 + kA_9)L_{j-1} + 9k\mu^{-1}\| \tau_{j+1} \|^2,$$

where

$$A_9 = A_7 + \mu^{-1}\eta^{-1}A_7.$$

We now assume k is so small that $2kA_6 \leqq 1$. Since

$$(1 - kA_6)^{-1} \leqq \exp(2kA_6),$$

it follows from (23) that

$$(24) \quad L_j \leqq e^{\alpha k}L_{j-1} + 9k\mu^{-1}\| \tau_{j+1} \|^2,$$

where

$$\alpha = 2A_6 + A_9.$$

If $\beta = 9\mu^{-1}A_2^2 T$, then (14) and (24) imply that

$$L_j \leqq e^{\alpha T}[L_0 + \beta(h^2 + k^2)^2].$$

Since $L_0 = 2\| z_1 \|_{0,D}^2 \leqq 2\eta^{-1}A_2^2(h^2 + k^2)^2$, in view of (9) and (15), we obtain the inequality

$$(25) \quad \| z_{j+1} \|_\infty^2 \leqq \| z_{j+1} \|_D^2 \leqq \eta^{-1}\| z_{j+1} \|_{j,D}^2 \leqq 4\eta^{-1}L_j \leqq A^2(h^2 + k^2)^2,$$

where

$$A^2 = 8\eta^{-1}e^{\alpha T}(A_2^2 + \beta).$$

Thus, we shall have $\| z_{j+1} \|_\infty \leqq k$ as soon as

$$A(1 + \lambda_-^{-2})k \leqq 1,$$

which finishes the induction step (note that our restrictions on the size of h and k are independent of u_j).

As a result of (25), we have also established the convergence theorem: $\| z_j \|_\infty \leqq A(h^2 + k^2)$.

An examination of the foregoing proof will show that the convergence theorem remains valid under the weaker assumption that the conditions in (2) hold only in some neighborhood of the solution w. This generalization shows, for example, that the convergence theorem holds when the difference scheme (4) is applied to positive solutions of

$$w^{1/2}\partial w/\partial t = \partial^2 w/\partial x^2.$$

Finally, we indicate that our results can be extended to equations of the form

$$b\left(x, t, w, \frac{\partial w}{\partial x}\right)\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(a(x, t, w)\frac{\partial w}{\partial x}\right) + c\left(x, t, w, \frac{\partial w}{\partial x}\right),$$

where $b \geqq \mu > 0$ and $a \geqq \eta > 0$. We put

$$\bar{a}(x, t\, v(x)) = a\left[x - \frac{h}{2}, t, \frac{v(x) + v(x - h)}{2}\right].$$

Then the difference scheme becomes $u_0 = \phi$,

$$u_1 = \phi + [k/b(x, 0, \phi, \phi')][(a(x, 0, \phi)\phi')' + f(x, 0, \phi, \phi')],$$

and, for $1 \leqq j < J$

$$b(x, jk, u_j, D_0u_j)(u_{j+1} - u_{j-1}) = 2kD_+(\bar{a}(x, jk, u_j)D_-\hat{u}_j) + 2kf(x, jk, u_j, D_0u_j).$$

The only major change in the convergence proof is that one proves by induction that $\| z_j \|_\infty \leqq k$ and

$$\max_{x-\{0\}} | D_-z_j | \leqq k.$$

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