and the corresponding value y_0,

$$F_p(y) = F_p(y_0) - I(x_0) + I(x) \tag{22}$$

is the final integral.

Autonetics
3370 Miraloma Avenue
Anaheim, California 92803

1. K. Pearson, *Tables for Statisticians and Biometricians*, Part I, Cambridge University Press for the Biometrika Trustees, 1914.
2. K. Pearson, *Tables of the Incomplete Gamma Function*, Cambridge University Press for the Biometrika Trustees, 1957.
3. E. B. Wilson & Margaret Hilferty, "The distribution of χ^2," *Proc. Nat. Acad. Sci.* v. 17, 1931.

Doppler Broadening Integrals*

By Van E. Wood, R. P. Kenan and M. L. Glasser

Asymptotic expansions in terms of Chebyshev polynomials of the integrals

$$I_n(x, v) = v^{1/2} \int_{-\infty}^{\infty} y^n (1 + y^2)^{-1} \exp[-v(y - x)^2]\, dy, \qquad n = 0, 1, \cdots$$

have recently been given by Thompson [1]. In this note, we wish to point out that these integrals can be expressed in terms of tabulated (and readily calculable) functions for any values of the parameters, a fact which is not mentioned in the literature we have seen on this subject. Specifically, we find

$$I_0(x, v) = \pi v^{1/2} \operatorname{Re}[\exp[v(1 + ix)^2] \operatorname{erfc}(v^{1/2}(1 + ix))],$$
$$I_1(x, v) = \pi v^{1/2} \operatorname{Im}[\exp[v(1 + ix)^2] \operatorname{erfc}(v^{1/2}(1 + ix))].$$

These results may be obtained by introducing the representation $(1 + y^2)^{-1} = \int_0^\infty e^{-z} \cos yz\, dz$, or by solving the coupled differential equations $dI_0/dx = 2v(I_1 - xI_0)$, $dI_1/dx = 2v(\pi^{1/2} - I_0 - xI_1)$, or by simply making appropriate changes of variable in some tabulated integrals [2]. Now a method for calculating error functions of complex argument using a rapidly converging infinite series has been described by Salzer [3]; hence the integrals can be obtained easily without using the asymptotic expansion.

By introducing the well-known asymptotic expansion for the co-error function, one may obtain in a very simple way the asymptotic expansions in Chebyshev polynomials

$$I_0 \sim \pi^{1/2} \sum_{m=0}^{\infty} (2m - 1)!!(-2v)^m (1 + x^2)^{-m-1/2} T_{2m+1}((1 + x^2)^{-1/2}),$$
$$I_1 \sim \pi^{1/2} \sum_{m=0}^{\infty} (2m - 1)!!(+2v)^m (1 + x^2)^{-m-1/2} T_{2m+1}(x/(1 + x^2)^{1/2}).$$

where

Received February 28, 1966.
* Work supported in part by United States Air Force Office of Scientific Research Grant number AF-AFOSR-260-63.

$$(2m-1)!! = 1\cdot 3\cdot 5 \cdots (2m-1), \qquad m > 0,$$
$$= 1 \qquad m = 0.$$

These expansions agree with those of Thompson [1] except that the factor of $\pi^{1/2}$ has been inadvertently omitted in his paper and there is a sign error in his expansion for I_1. (Apparently the author intended to include a factor $\pi^{-1/2}$ in defining the integrals, as the reciprocal factor is otherwise necessary in all the other expressions for the integrals in the paper. There are also other minor misprints in Eq. (2), the equation preceding it, and in the first, second, third, and fifth equations following Eq. (1).)

Battelle Memorial Institute
Columbus Laboratories
Columbus, Ohio 43201

1. G. T. Thompson, "The asymptotic expansion of the integrals psi and chi in terms of Tchebycheff polynomials," *Math. Comp.*, v. 19, 1965, pp. 661–663.
2. M. Abramowitz & I. A. Stegun, (Editors), *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series, No. 55, U. S. Government Printing Office, Washington, D. C., 1964, pp. 302–303.
3. H. E. Salzer, "Formulas for calculating the error function of a complex variable," *MTAC*, v. 5, 1951, pp. 67–70. MR **13**, 989, 1140.

Singular and Invariant Matrices Under the *QR* Transformation*

By Beresford Parlett

0. Introduction. The above two classes of matrix are usually ignored in discussions of the QR algorithm [1], [3], [4]. Some familiarity with the algorithm is assumed.

There are good reasons for this neglect. Firstly the algorithm is not well defined for singular matrices and thus its behavior is difficult to describe. Secondly the problem of describing all matrices which are left invariant under the QR transformation is in general rather difficult.

The purpose of this note is to point out that with a preliminary reduction to Hessenberg form both difficulties disappear. We show first that singular matrices reveal their zero eigenvalues, one per iteration. Secondly we describe all matrices which are invariant.

A given square matrix A (real or complex) may be reduced to Hessenberg form A_1 ($a_{ij} = 0$, $i > j + 1$) in a variety of ways. If any subdiagonal elements $a_{i+1,i}$ vanish then A_1 is said to be reduced and may be partitioned appropriately as

$$(1) \qquad A_1 = \begin{pmatrix} H_1 & H_{12} & \cdot & H_{1m} \\ 0 & H_2 & \cdot & H_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ & & & H_m \end{pmatrix}$$

Received March 14, 1966. Revised May 5, 1966.

* This work was begun while the author was a summer visitor at the Argonne National Laboratory, Argonne, Illinois. It was completed under contract NSF GP-4436 at the University of California, Berkeley.