

Quadratures with Remainders of Minimum Norm. I *

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1. Introduction. One approach to the problem of numerical integration on contours in the complex plane was discussed in a previous paper [1]. This paper is an application of that work to the quadrature problem $\int_{-1}^1 f(x) \cong \sum_{k=1}^n A_k f(z_k)$. For a fixed n , $\{A_k\}_{k=1}^n$ and $\{z_k\}_{k=1}^n$ form a set of $2n$ variables. Let $R_n(f) = \int_{-1}^1 f(z) dz - \sum_{k=1}^n A_k f(z_k)$. Then R_n is a linear functional defined on a certain function space X . In the cases to be discussed, X is a Hilbert space. Suppose that $\{P_m\}_{m=0}^\infty$ is a closed set of functions in X . Many of the standard quadrature methods amount to specifying certain side conditions and, subject to these, choosing the A_k and z_k so that $R_n^{-1}(0) = \{f \in X : R_n(f) = 0\}$ contains a maximal number of the P_m . If P_0, P_1, \dots, P_s are in $R_n^{-1}(0)$ and P_{s+1} is not in $R_n^{-1}(0)$, then the quadrature is said to have precision s . Thus, for example, if $X = C[a, b]$ and $P_m(x) = x^m$, $m = 0, 1, \dots$, and the side conditions are that the z_k are specified beforehand, then choosing the A_k so that $R_n^{-1}(0)$ contains a maximal number of the $P_m(x)$ amounts to the usual interpolatory quadrature procedure.

The procedure used in this paper is to minimize the norm of a certain error functional related to R_n . Specifically, suppose that $R_n(f) = R_n^*(f^{(r)})$, where $f^{(r)}(z)$ denotes the r th derivative of f , and R_n^* is a linear functional defined on the space of r th derivatives of functions in X . Then $|R_n(f)| = |R_n^*(f^{(r)})| \leq \|R_n^*\| \cdot \|f^{(r)}\|$, and the problem is that of minimizing $\|R_n^*\|$ by an appropriate choice of the A_k and z_k . It has been shown [1] that minimizing with respect to the A_k (for fixed z_k) yields systems of linear equations to be solved; minimizing with respect to the z_k yields systems of nonlinear equations. In this paper, the z_k are assumed given beforehand and only the A_k are treated as variables. In a later paper, both the linear and nonlinear cases will be considered.

2. The Equations to be Solved. In this paper we consider three sets of quadratures. For the first two sets, the space X is $L^2(E_\rho)$, the L^2 completion of $H(E_\rho)$, where $H(E_\rho) = \{f : f \text{ is analytic on the ellipse } E_\rho \text{ with semimajor axis } a, \text{ semiminor axis } b = (a^2 - 1)^{1/2} \text{ and } \rho = (a + b)^2\}$ [2]. The "double integral norm" is obtained from the inner product

$$(f, g) = \iint_{E_\rho} f(z) \overline{g(z)} \, dx dy.$$

For the first set of quadratures, $\{P_m(z)\}_{m=0}^\infty$ is defined by

$$P_m(z) = 2(m + 1)^{1/2} [\pi(\rho^{m+1} - \rho^{-m-1})]^{-1/2} U_m(z),$$

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where $U_m(z) = (1 - z^2)^{-1/2} \sin [(m + 1) \arccos(z)]$, $m = 0, 1, \dots$. In fact, the set $\{P_m(z)\}_{m=0}^\infty$ forms a closed orthonormal set in $L^2(E_\rho)$. No polynomial precision requirements are imposed for this set of quadratures so that $R_n = R_n^*$ and $\|R_n\|^2 = \sum_{m=0}^\infty |R_n(P_m)|^2$. If the z_k are given in $[-1, 1]$, then the A_k are real and denoted by a_k . The resulting linear system of equations in the variables a_k is as follows:

$$(1) \quad \sum_{j=1}^n a_j \left[\sum_{m=0}^\infty \alpha(m, \rho) U_m(z_j) U_m(z_k) \right] = \sum_{m=0}^\infty \alpha(m, \rho) \gamma(m) U_m(z_k), \quad k = 1, \dots, n,$$

where $\alpha(m, \rho) = 4(m + 1)[\pi(\rho^{m+1} - \rho^{-m-1})]^{-1}$, and

$$\gamma(m) = (m + 1)^{-1}[1 + (-1)^m], \quad m = 0, 1, \dots$$

In matrix notation, (1) can be written as $Ta = h$, where $T = (t_{jk})$, $t_{jk} = \sum_{m=0}^\infty \alpha(m, \rho) U_m(z_j) U_m(z_k)$, $a = (a_1, \dots, a_n)^T$, and $h = (h_1, h_2, \dots, h_n)^T$ where $h_k = \sum_{m=0}^\infty \alpha(m, \rho) \gamma(m) U_m(z_k)$.

For the second set of quadratures, $\{P_m(z)\}_{m=0}^\infty$ is defined by $P_m'(z) = 2(m + 1)^{1/2}[\pi(\rho^{m+1} - \rho^{-m-1})]^{-1/2} U_m(z)$, where $U_m(z)$ is as above. For this set, precision is required for constant functions so that $\|R_n^*\|^2 = \sum_{m=0}^\infty |R_n(Q_m)|^2$, where $Q_m(z) = P_m'(z)$. Assuming that the z_k are given in $[-1, 1]$, the resulting system of equations to be solved is the following:

$$(2) \quad \sum_{j=1}^n a_j \left[2 \sum_{m=0}^\infty \alpha(m, \rho) T_{m+1}(z_j) T_{m+1}(z_k) \right] = 2 \sum_{m=0}^\infty \alpha(m, \rho) \beta(m) T_{m+1}(z_k) - \lambda_1, \quad k = 1, \dots, n,$$

where λ_1 is a parameter, $T_{m+1}(z) = \cos [(m + 1) \arccos(z)]$, the $(m + 1)$ st Tchebycheff polynomial of the first kind, a_j and $\alpha(m, \rho)$ are as above, and $\beta(0) = 0$, $\beta(m) = [1 - (m + 1)^2]^{-1}[1 + (-1)^{m+1}]$, $m = 1, 2, \dots$. The system of equations (2) can be written in matrix form as $Ta = h - \lambda$, where

$$\begin{aligned} T &= (t_{jk}), \\ t_{jk} &= 2 \sum_{m=0}^\infty \alpha(m, \rho) T_{m+1}(z_j) T_{m+1}(z_k), \\ a &= (a_1, \dots, a_n)^T, \\ h &= (h_1, \dots, h_n)^T, \\ h_k &= 2 \sum_{m=0}^\infty \alpha(m, \rho) \beta(m) T_{m+1}(z_k), \end{aligned}$$

and

$$\lambda = (\lambda_1, \lambda_1, \dots, \lambda_1)^T.$$

In the third set of quadratures, the space X is the L^2 completion of $H([-1, 1])$, where $H([-1, 1]) = \{f: f \text{ is analytic on } [-1, 1]\}$. The "line integral norm" is obtained from the inner product

$$(f, g) = \int_{-1}^1 (1 - z^2)^{1/2} f(z) \overline{g(z)} |dz|,$$

where $|dz|$ denotes arc length. The L^2 completion of $H([-1, 1])$ is with respect to the norm induced by this inner product. The Tchebycheff polynomials of the second kind form a closed orthogonal set in this space. Precision is required for constant functions so that $R_n(f) = R_n^*(f')$ where f' is the derivative of f . This is the case $r = 1$ in Section 1. The resulting system of linear equations to be solved is the following:

$$(3) \quad \sum_{j=1}^n a_j \left[2 \sum_{m=0}^{\infty} \delta(m) T_{m+1}(z_j) T_{m+1}(z_k) \right] = 2 \sum_{m=0}^{\infty} \delta(m) \beta(m) T_{m+1}(z_k) - \lambda_1, \\ k = 1, \dots, n,$$

where $\beta(m)$ and $T_{m+1}(z)$ are defined above, $\delta(m) = (m + 1)^{-2} [\lambda(m)]^2$, $\lambda(m) = 2 \cdot 1 \cdot 3 \cdot 5 \cdots (2m + 1) / 2 \cdot 4 \cdots (2m + 2)$, and λ_1 is a parameter. As before, system (3) can be written in matrix form as $Ta = h - \lambda_1$.

3. Description of the Computations. The elements of the matrix T and the vector h are infinite sums and hence must be approximated by finite sums. The approximating finite sums were obtained by fixing a positive integer N and summing the series until N consecutive partial sums were identical within the double precision range of the computer (16 digits). For the double integral norm, N was fixed at 25, and the number of terms in the approximating series varied from 1050 to 35 as a varied from 1.0001 to 5.0. For the line integral norm, N was fixed at 500 and the number of terms in the approximating series averaged about 350,000. The slow rate of convergence of the series for the line integral norm made it impractical to calculate T using more than ten abscissas. Using the IBM 7044 it requires approximately 9 hours to compute and sum 10,000 terms of the series in the matrix T for the 16 Gauss abscissas.

A direct method was used to calculate T^{-1} . Then Newton's Method was used to improve the calculated T^{-1} . If D_0 is the approximation to T^{-1} , calculated by a direct method, then $D_1 = D_0(2I - TD_0)$ is the first iterate, with error measured by $E_1 = I - TD_1$. The iteration was terminated after m steps when all the elements of E_m were less than 10^{-14} in absolute value.

For the double integral norm calculations, the matrix T becomes unstable with increasing values of a . For example, the determinant of T , using Gauss 7 point abscissas and $a \geq 1.75$, becomes smaller than 10^{-27} .

For this reason we were unable to obtain values for the quadrature weights and remainders with predictable reliability for some sets of base points for relatively large values of a . Hence, these are omitted from the following tables.

The linear system of equations is solved by the Gauss-Seidel method [4]. Since T is real, positive definite, symmetric and has positive diagonal terms, the Gauss-Seidel method is known to converge.

The approximations to $\|R_n^*\|$ and $\|R_n\|$ were calculated in the same manner as the approximations to the elements of the matrix T . Lee, Sun, and Lo [5] have calculated $\|R_n\|$ using known quadrature weights and we have verified their results as a check on our calculations.

4. Tables. The following set of tables (tables 1-5) lists the quadrature weights and the norms of the remainders that were calculated using known abscissas. These

TABLE 1
 Double Integral Norm—No Polynomial Precision—Quadrature Weights

<i>a</i>	Simpson	Gauss 3 Pt	Gauss 4 Pt	Gauss 5 Pt
1.0300	0.48361078436 (-01) 0.62190138805 (00)	0.37908748908 (00) 0.60611290281 (00)	0.28412793600 (00) 0.53597916953 (00)	0.21381861721 (00) 0.43609677948 (00) 0.51766807514 (00)
1.1000	0.15020278474 (00) 0.10802156271 (01)	0.52537861849 (00) 0.84982939513 (00)	0.34260959376 (00) 0.64587820577 (00)	0.23593149740 (00) 0.47791462274 (00) 0.56743066540 (00)
1.3000	0.28075945513 (00) 0.13516110684 (01)	0.55420054206 (00) 0.88882629800 (00)	0.34776511657 (00) 0.65214610257 (00)	0.23691694967 (00) 0.47864229251 (00) 0.56887084058 (00)
1.4000	0.30116436175 (00) 0.13595789490 (01)	0.55507595464 (00) 0.88909969008 (00)	0.34783258104 (00) 0.65215202334 (00)	0.23692462249 (00) 0.47863287190 (00) 0.56888382173 (00)
1.5000	0.31218420402 (00) 0.13573247426 (01)	0.55535361671 (00) 0.88905236490 (00)	0.34784804875 (00) 0.65214857350 (00)	0.23692622672 (00) 0.47863008978 (00) 0.56888718883 (00)
1.7500	0.32420798817 (00) 0.13476570582 (01)	0.55551891264 (00) 0.88893916235 (00)	0.34785423425 (00) 0.65214561823 (00)	0.23692682899 (00) 0.47862881183 (00) 0.56888871481 (00)
2.0000	0.32860624413 (00) 0.13416669524 (01)	0.55554580970 (00) 0.88890491738 (00)	0.34785475555 (00) 0.65214523265 (00)	0.23692687701 (00) 0.47862869197 (00) 0.56888886191 (00)
3.0000	0.33255885909 (00) 0.13348503035 (01)	0.55555529603 (00) 0.8888939143 (00)	0.34785484472 (00) 0.65214515527 (00)	0.23692690750 (00) 0.47862862039 (00) 0.56888895324 (00)
4.0000	0.33310159153 (00) 0.13337939281 (01)	0.55555553234 (00) 0.88888893486 (00)	0.34785484511 (00) 0.65214515489 (00)	0.23692747878 (00) 0.47862698456 (00) 0.56889107333 (00)

TABLE 2
Double Integral Norm—Precision for Constants—Quadrature Weights

<i>a</i>	Simpson	Gauss 3 Pt	Gauss 4 Pt	Gauss 5 Pt
1.0300	0.33985563522 (00) 0.13202887296 (01)	0.56328993906 (00) 0.87342012189 (00)	0.35090717201 (00) 0.64909282799 (00)	0.23808296455 (00) 0.47764692477 (00) 0.56854022136 (00)
1.1000	0.33476544379 (00) 0.13304691124 (01)	0.55741546447 (00) 0.88516907106 (00)	0.34815569111 (00) 0.65184430889 (00)	0.23698875159 (00) 0.4785235395 (00) 0.56891778892 (00)
1.3000	0.33345272651 (00) 0.13330945470 (01)	0.55572211884 (00) 0.88855576232 (00)	0.34786178021 (00) 0.65213821979 (00)	0.23692764050 (00) 0.47862717357 (00) 0.56889037187 (00)
1.4000	0.33338267158 (00) 0.13332346568 (01)	0.55562516076 (00) 0.88874967848 (00)	0.34785667116 (00) 0.65214332884 (00)	0.23692705643 (00) 0.47862828845 (00) 0.56888931024 (00)
1.5000	0.33335634036 (00) 0.13332873193 (01)	0.55558822358 (00) 0.88892355284 (00)	0.34785542397 (00) 0.65214457603 (00)	0.23692693486 (00) 0.47862855123 (00) 0.56888902783 (00)
1.7500	0.33333811335 (00) 0.13333237733 (01)	0.55556239417 (00) 0.88887521166 (00)	0.34785489571 (00) 0.65214510429 (00)	0.23692688463 (00) 0.47862866391 (00) 0.56888890292 (00)
2.0000	0.33333468286 (00) 0.13333306343 (01)	0.55555749222 (00) 0.88888501556 (00)	0.34785485311 (00) 0.65214514689 (00)	0.23692688621 (00) 0.47862866693 (00) 0.56888889372 (00)
3.0000	0.3333336862 (00) 0.1333332628 (01)	0.55555560570 (00) 0.8888878859 (00)	0.34785484062 (00) 0.65214515938 (00)	0.23692711557 (00) 0.47862800943 (00) 0.56888974999 (00)
4.0000	0.3333333635 (00) 0.1333333273 (01)	0.5555555981 (00) 0.8888889037 (00)	0.34785484447 (00) 0.65214515553 (00)	

TABLE 3
Double Integral Norm—No Polynomial Precision $\|R_n\|_{L_p}$

a	Simpson	Gauss 3 Pt	Gauss 4 Pt	Gauss 5 Pt
1.03	0.1907241070 (01)	0.1382887314 (01)	0.1035294859 (01)	0.7362638037 (00)
1.10	0.9274917925 (00)	0.3845184443 (00)	0.1850910254 (00)	0.8541217118 (-01)
1.30	0.2348739814 (00)	0.4382165030 (-01)	0.1112196675 (-01)	0.2735533155 (-02)
1.40	0.1377645492 (00)	0.2021109939 (-01)	0.4110587269 (-02)	0.8104460513 (-03)
1.50	0.8628845236 (-01)	0.1036395250 (-01)	0.1741600505 (-02)	0.2837324786 (-03)
1.75	0.3260704862 (-01)	0.2629888280 (-02)	0.2973710988 (-03)	0.3271112192 (-04)
2.00	0.1482910137 (-01)	0.8662381058 (-03)	0.7163719096 (-04)	0.5743495614 (-05)
3.00	0.1599887948 (-02)	0.3822805818 (-04)	0.1286249079 (-05)	0.4261136510 (-07)
4.00	0.3558590379 (-03)	0.4658675058 (-05)	0.8657541858 (-07)	0.1707253302 (-08)

TABLE 4
Double Integral Norm—Precision for Constants $\|R_n^*\|_{L_p}$

a	Simpson	Gauss 3 Pt	Gauss 4 Pt	Gauss 5 Pt
1.03	0.4778521648 (00)	0.1837050022 (00)	0.9866281887 (-01)	0.5460451818 (-01)
1.10	0.2065191174 (00)	0.5090701262 (-01)	0.1844101181 (-01)	0.6861281113 (-02)
1.30	0.5851012530 (-01)	0.7375292963 (-02)	0.1429282604 (-02)	0.2844313295 (-03)
1.40	0.3756548466 (-01)	0.3767137950 (-02)	0.5851762843 (-03)	0.9334986464 (-04)
1.50	0.2563768033 (-01)	0.2114979918 (-02)	0.2714648030 (-03)	0.3578247266 (-04)
1.75	0.1168282634 (-01)	0.6472781196 (-03)	0.5609504949 (-04)	0.4992330298 (-05)
2.00	0.6205219059 (-02)	0.2500425662 (-03)	0.1579374804 (-04)	0.1024470128 (-05)
3.00	0.1043010341 (-02)	0.1719734827 (-04)	0.4453774465 (-06)	0.1187412312 (-07)
4.00	0.3132777380 (-03)	0.2830101094 (-05)	0.4016920187 (-07)	

TABLE 5
Line Integral Norm—Precision for Constants

Abscissas	Weights	$\ R_n^*\ $
Trapezoid	1.0000000000	1.0650733423
Simpson	0.34166233122 1.31667533756	0.3225321140
Weddle	0.11442868201 0.40254092837 0.31437632017 0.33730813889	0.0681382406
Gauss 2 pt.	1.0000000000	0.2550563770
Gauss 3 pt	0.56573797607 0.86852404786	0.1498311203
Gauss 4 pt	0.35402340648 0.64597659352	0.1020021252
Gauss 5 pt	0.24103044858 0.47512812123 0.56768286040	0.0752714571
Gauss 7 pt	0.13156595754 0.27822895364 0.38147838891 0.41745338982	0.0471841912
Gauss 10 pt	0.06074063246 0.28496489128 -0.48032018629 1.60088303881 -0.46626837611	0.3475518407

quantities were calculated for 22 values of a ranging from 1.0001 to 5.0 for abscissas corresponding to the following quadrature rules: trapezoid, Simpson, Gauss 2, 3, 4, 5, 7, 10, and 16 point rules, and Weddle. For the trapezoidal rule and the Gauss 2-point rule, the weights are 1.0. The quantities in the above list that are not in the tables in this paper are in the Unpublished Mathematical Tables File.

The quadrature weights are labeled from left to right on $[-1, 1]$. Since symmetric abscissas yield symmetric weights, the duplications are omitted from the tables. The numbers are expressed as floating point decimals with the number in parentheses being the power of ten.

5. Examples and Use of Tables. The table following the examples (table 6) gives comparisons, for specific functions, of our quadratures with various known quadratures. For the examples given, the same abscissas were used for both the known quadratures and our quadratures. The numbers in parenthesis under the double integral norms indicate the values of a from which the quadrature weights were computed.

As a specific example for using the minimum norm remainders listed in the previous tables, consider the function $f(z) = e^{z^2}$.

(a) Double integral norm—no polynomial precision:

$$|R_n(f)| \leq \|R_n\| \cdot \|f\|, \text{ where } \|R_n\| \text{ is tabulated in the tables and}$$

TABLE 6
Table of Examples

Function	Interval of Integration	Abscissas	Error Using Known Weights	Error Using Line Integral Norm Weights	Error Using Double Integral Norm Weights—0 Precision	Error Using Double Integral Norm Weights—No Precision
$f_1(x) = x^{20}$	$[-1, 1]$	Simpson	0.01172	0.01353	(2.0) 0.01172	(1.25) 0.00572 (1.30) 0.00019 (1.40) 0.00408 (1.50) 0.00682 (1.75) 0.00967 (2.0) 0.01068
$f_1(x)$	$[-1, 1]$	Gauss 4 pt.	0.3569 (-06)	0.5167 (-06)	(2.5) 0.3586 (-06) (3.0) 0.2755 (-06)	(3.0) 0.3507 (-06) (4.0) 0.1421 (-06)
$f_2(x) = (4x^2 + 2x)/(x^4 + x^2 + 2)$	$[0, 1]$	Simpson	-0.2380 (-03)	-0.1579 (-03)	(1.01) 0.1105 (-03) (1.02) 0.1518 (-03)	(5.0) 0.2391 (-03)
$f_2(x)$	$[0, 1]$	Gauss 3 pt.	0.4709 (-03)	-0.3301 (-03)	(1.04) -0.3700 (-05)	(5.0) 0.4709 (-05)
$f_2(x)$	$[0, 1]$	Gauss 5 pt.	0.1645 (-05)	-0.2213 (-03)	(1.10) -0.6202 (-06)	(2.0) 0.1645 (-05) (2.5) 0.1644 (-05)
$f_3(x) = \begin{cases} 1 & -1 \leq x \leq -\frac{1}{2} \\ 4 & -\frac{1}{2} \leq x \leq 0 \\ 2 & 0 < x \leq \frac{1}{2} \\ -4 & \frac{1}{2} < x \leq 1 \end{cases}$	$[-1, 1]$	Simpson	-0.4167	-0.4271	(1.5) -0.4167	(1.15) 0.0217 (1.20) -0.1332 (1.25) -0.2274 (1.75) 0.3937 (-12)
$f_3(x)$	$[-1, 1]$	Gauss 7 pt.	0.0	0.0	(all α) 0.0	
$f_3(x)$	$[-1, 1]$	Weddle	-0.1250	-0.1350	(1.75) -0.1229	(1.05) -0.0182 (1.10) -0.0998
$f_4(x) = \begin{cases} 3 & -1 \leq x \leq -\frac{1}{2} \\ 8 & -\frac{1}{2} < x \leq -\frac{1}{3} \\ 7 & -\frac{1}{3} < x \leq -\frac{1}{4} \\ 4 & -\frac{1}{4} < x \leq -\frac{1}{5} \\ 6 & -\frac{1}{5} < x \leq -\frac{1}{6} \\ 13 & -\frac{1}{6} < x \leq 1 \end{cases}$	$[-1, 1]$	Gauss 3 pt.	-0.1917	-0.1878	(1.04) -0.1894 (1.25) -0.1916 (1.40) -0.1317	(1.15) 0.0451 (1.20) -0.0987 (1.10) -0.0976 (1.15) -0.1265
$f_4(x)$	$[-1, 1]$	Gauss 5 pt.	-0.1317	-0.1314		

$$\|f\| = \left\{ \iint_{E_p} |f(z)|^2 dx dy \right\}^{1/2} \leq M \left\{ \iint_{E_p} dx dy \right\}^{1/2} = M(\pi ab)^{1/2},$$

where $M = \max_{z \in E_p} |f(z)|$.

For $f(z) = e^{z^2}$, $M = \max_{z \in E_p} |e^{z^2}| = \max_{z \in E_p} |e^{x^2 - y^2}| = e^{a^2}$.

Therefore, $|R_n(f)| \leq \|R_n^* \| \cdot e^{a^2} (\pi ab)^{1/2} = \|R_n^* \| \cdot e^a [\pi a(a^2 - 1)^{1/2}]^{1/2}$.

The following minimum values of $\|R_n^* \| \cdot e^{a^2} (\pi ab)^{1/2}$ are obtained for the indicated abscissas from the values of a used in the preceding tables.

<i>Abscissas</i>	a	$\ R_n^* \ \cdot e^{a^2} [\pi a(a^2 - 1)^{1/2}]^{1/2}$
Gauss 2 pt.	1.50	1.26993
Simpson	1.50	1.87016
Gauss 4 pt.	2.00	0.01290
Gauss 7 pt.	2.00	0.63077×10^{-5}
Gauss 10 pt.	1.50	0.58202×10^{-6}

(b) Double integral norm—precision for constants:

$|R_n(f)| \leq \|R_n^* \| \cdot \|f'\|$, where $\|R_n^* \|$ is tabulated and

$$\|f'\| = \left\{ \iint_{E_p} |f'(z)|^2 dx dy \right\}^{1/2} \leq M'(\pi ab)^{1/2},$$

where $M' = \max_{z \in E_p} |f'(z)|$.

For $f(z) = e^{z^2}$, $M' = \max_{z \in E_p} |2ze^{z^2}| = \max_{z \in E_p} 2(x^2 + y^2)^{1/2} e^{x^2 - y^2}$, therefore, $M' = 2ae^{a^2}$, and $\|f'\| \leq 2ae^{a^2} (\pi ab)^{1/2}$.

$$|R_n(f)| \leq \|R_n^* \| \cdot 2ae^{a^2} [\pi a(a^2 - 1)^{1/2}]^{1/2}.$$

The following minimum values of $\|R_n^* \| \cdot 2ae^{a^2} [\pi a(a^2 - 1)^{1/2}]^{1/2}$ are obtained for the indicated abscissas from the values of a used in the preceding tables.

<i>Abscissas</i>	a	$\ R_n^* \ \cdot 2ae^{a^2} [\pi a(a^2 - 1)^{1/2}]^{1/2}$
Gauss 3 pt.	1.75	0.13611
Simpson	1.30	1.51852
Gauss 5 pt.	2.00	0.73809×10^{-3}
Weddle	2.00	0.12802×10^{-1}

(c) Line integral norm—precision for constants:

$|R_n(f)| \leq \|R_n^* \| \cdot \|f'\|$, where $\|R_n^* \|$ is tabulated and

$$\|f'\| = \left\{ \int_{-1}^1 (1 - z^2)^{1/2} |f'(z)|^2 dz \right\}^{1/2},$$

where $|dz|$ is arc length.

$$\|f'\| = \left\{ \int_{-1}^1 (1 - z^2)^{1/2} |f'(z)|^2 dz \right\}^{1/2} = \left\{ \int_{-1}^1 (1 - x^2)^{1/2} |f'(x)|^2 dx \right\}^{1/2}$$

for the contour $[-1, 1]$. If we let $M = \max_{x \in [-1, 1]} [(1 - x^2)^{1/2} |f'(x)|]^2$, then

$$\|f'\| \leq \sqrt{M} \left\{ \int_{-1}^1 dx \right\}^{1/2} = \sqrt{2M}.$$

For $f(z) = e^{z^2}$,

$$\begin{aligned} \|f'(z)\| &= \left\{ \int_{-1}^1 (1-x^2)^{1/2} |2ze^{z^2}|^2 dz \right\}^{1/2} \\ &= 2 \left\{ \int_{-1}^1 (1-x^2)^{1/2} x^2 e^{2x^2} dx \right\}^{1/2} \leq 2\sqrt{e^2} \cdot \sqrt{2} = 2e\sqrt{2}. \end{aligned}$$

Therefore,

$$|R_n(f)| \leq \|R_n^*\| \cdot 2e\sqrt{2}.$$

The following values for $\|R_n^*\| \cdot 2e\sqrt{2}$ are obtained for the indicated abscissas.

Abscissa	$\ R_n^*\ \cdot 2e\sqrt{2}$
Simpson	2.47978
Weddle	0.52388
Gauss 3 pt.	1.15197
Gauss 5 pt.	0.57872
Gauss 7 pt.	0.36277

6. Conclusions. It appears as if the double integral norm quadratures with no polynomial precision will be the most practical of the three sets of quadratures. The line integral norm quadratures involved series which converge very slowly, as noted in Section 3. Hence these quadratures would probably be too expensive to compute for most practical purposes.

For the examples calculated, the results of using the double integral norm quadratures with precision for constants did not seem significantly different from the results obtained using Gaussian quadratures. However, most of these comparisons were made using Gaussian abscissas and hence the Gaussian quadratures probably appear better than they might if compared with quadratures involving other abscissas.

It might be noted that if the z_k are given, but are complex, then the system of equations to be solved doubles in size. Also, as mentioned above, it is planned to treat the z_k as variables as well as the A_k .

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