Practical L^p Polynomial Approximation

By F. D. Burgoyne

Let p denote a positive integer, and let $P_{pq}(x)$ denote the polynomial of degree q with coefficient of x^q unity which gives the least value when the pth power of its modulus is integrated over $(-1, 1)$. The extension of what is given below to an interval other than $(-1, 1)$ is trivial. The existence and uniqueness of $P_{pq}(x)$ follows from the general theory of L^p approximation, see, for example, [1].

If we write $P_{pq}(x) = (x - x_1) \cdots (x - x_q)$, we may deduce from elementary considerations that the x_r are real, interior to $(-1, 1)$, symmetrically placed with respect to $x = 0$ and distinct.

Clearly $P_{1q} = 2^{-q}U_q(x)$, where $U_q(x)$ is the Chebyshev polynomial of the second kind of degree q, and $P_{2q}(x) = \{2^q(q!)^2/(2q)!\}P_q(x)$, where $P_q(x)$ is the Legendre polynomial of degree q. Since $U_q(x)$ and $P_q(x)$ are the ultraspherical polynomials $P_q^{(1)}(x)$ and $P_q^{(1/2)}(x)$ respectively, it may be conjectured that $P_{pq}(x) = k_qP_q^{(1/p)}(x)$; however this is disproved by the fact that the zeros of $P_{42}(x)$ are about ± 0.629, while those of $P_2^{(1/4)}(x)$ are about ± 0.632.

Below we tabulate the positive zeros of $P_{pq}(x)$ to five decimal places for $p, q = 2(1)7$ (table 1). To each positive zero there corresponds a negative zero of equal magnitude, and, for odd q, $x = 0$ is also a zero. The coefficients of $P_{pq}(x)$ may be obtained from the given zeros. We also tabulate $L^p(P_{pq})$, where $L^p(f)$ denotes $\{\int_{-1}^1 |f(x)|^p\,dx\}^{1/p}$. The case $p = 1$ was not included since we have $P_{1q}(x) = (x - \cos(\pi/(q+1))) \cdots (x - \cos(q\pi/(q+1)))$ and $L^1(P_{1q}) = 2^{1-q}$. On the other hand the case $p = 2$ was included for convenience and purposes of comparison, although $P_{2q}(x)$ is essentially a well-known polynomial. Note that, for all p, $P_{p0}(x) = 1$ and $P_{p1}(x) = x$.

The zeros x_r were evaluated as follows. Suppose first that q is even with $Q = \frac{1}{2}q$, then we may take $P_{pq}(x) = (x^2 - x_1^2) \cdots (x^2 - x_Q^2)$ where $0 < x_1 < \cdots < x_Q < 1$. In particular, for $p = 1$ we have $x_r = \cos((Q - r + 1)\pi/(q + 1))$. Since $\int_{-1}^1 |P_{pq}(x)|^p dx$ is a minimum we obtain

$$\int_0^1 \{|x^2 - x_1^2|^p \cdots |x^2 - x_Q^2|^p/(x^2 - x_r^2)\}\,dx = 0.$$

We now solve these equations iteratively by taking $x_{r0}(p) = x_r(p-1)$ (where the notation shows the dependence on p) and $x^2_{r(s+1)} = x^2_{rs} + \epsilon_{rs}$, where $pb_{r1s}\epsilon_{1r} + \cdots + pb_{rQs}\epsilon_{Qs} - b_{rrs}\epsilon_{rs} = a_{rs}$,

$$a_{rs} = \int_0^1 \{|x^2 - x^2_{1s}|^p \cdots |x^2 - x^2_{Qs}|^p/(x^2 - x^2_{rs})\}\,dx$$

and

$$b_{rts} = \int_0^1 \{|x^2 - x^2_{1s}|^p \cdots |x^2 - x^2_{Qs}|^p/(x^2 - x^2_{rs})(x^2 - {}^2_{ts})\}\,dx.$$

This amounts to ignoring powers higher than the first in the ϵ's, and the convergence is quite rapid. If q is odd and we define $Q = \frac{1}{2}(q - 1)$ the above expressions are unaltered except that each integrand contains the term x^p. The computation was carried out on the University of London Atlas.

Received May 16, 1966. Revised July 31, 1966.

TABLE 1

p	q	Positive zeros of $P_{pq}(x)$			$L^p(P_{pq})$
2	2	0.57735			0.42164
3	2	0.61047			0.41318
4	2	0.62921			0.41476
5	2	0.64140			0.41855
6	2	0.65003			0.42271
7	2	0.65648			0.42670
2	3	0.77460			0.21381
3	3	0.80053			0.21003
4	3	0.81444			0.21086
5	3	0.82319			0.21268
6	3	0.82923			0.21464
7	3	0.83368			0.21652
2	4	0.33998	0.86114		0.10775
3	4	0.35223	0.87991		0.10596
4	4	0.35886	0.88968		0.10637
5	4	0.36304	0.89571		0.10725
6	4	0.36592	0.89983		0.10819
7	4	0.36804	0.90283		0.10909
2	5	0.53847	0.90618		0.05415
3	5	0.55310	0.92004		0.05328
4	5	0.56087	0.92712		0.05348
5	5	0.56571	0.93144		0.05391
6	5	0.56903	0.93437		0.05437
7	5	0.57146	0.93649		0.05480
2	6	0.23862	0.66121	0.93247	0.02717
3	6	0.24470	0.67513	0.94302	0.02674
4	6	0.24792	0.68244	0.94834	0.02684
5	6	0.24991	0.68695	0.95157	0.02705
6	6	0.25128	0.69003	0.95374	0.02728
7	6	0.25227	0.69227	0.95531	0.02749
2	7	0.40585	0.74153	0.94911	0.01362
3	7	0.41442	0.75396	0.95738	0.01341
4	7	0.41891	0.76041	0.96150	0.01346
5	7	0.42169	0.76438	0.96399	0.01356
6	7	0.42358	0.76708	0.96566	0.01367
7	7	0.42495	0.76903	0.96687	0.01378

The tables enable practical L^p polynomial approximation to be performed approximately in a variety of ways. Thus, to obtain an L^p polynomial approximation of degree $q - 1$ over $(-1, 1)$ to a function $f(x)$, we may collocate at the zeros of $P_{pq}(x)$, and so get an error of $(x - x_1) \cdots (x - x_q)f^{(q)}(\xi)/q! = P_{pq}(x)f^{(q)}(\xi)/q!$, where ξ belongs to $(-1, 1)$, assuming the existence and continuity of $f^{(q)}(x)$ over the interval. In the L^p sense this error is the least possible if $f^{(q)}(x)$ is constant over the interval, and in other cases the method may be assumed to give a good approximation to the desired polynomial.* Alternatively, if $f(x)$ possesses a power series expansion we may truncate the latter at the term involving x^Q, where $Q \geqq q$, and rearrange the resulting polynomial in terms of $P_{p0}(x), P_{p1}(x), \cdots, P_{pQ}(x)$ and "economize" (in an analogous way to Lanczos' economization procedure [2]) to a polynomial of degree $q - 1$ expressed in terms of $P_{p0}(x), P_{p1}(x), \cdots, P_{p(q-1)}(x)$. In this case an upper bound for the error in the L^p sense is given by the sum of the pth root of the integrals of the pth power of the modulus of the neglected terms. In both methods the L^p error may be estimated by using the tabulated values of $L^p(P_{pq})$. In special cases it may be convenient to use other methods to approximate $f(x)$. Thus, if $f(x)$ satisfies a suitable differential equation, we could use a procedure rather like that of Clenshaw [3] for Chebyshev-type approximation. A similar technique may be used if $f(x)$ is a rational function.

Finally we consider a simple numerical example in which $f(x) = \log(\frac{3}{2} + x/2)$, $p = 3$ and $q = 3$. Starting with the approximation $f(x) \doteq \log \frac{3}{2} + x/3 - x^2/18 + x^3/81$ a simple iterative method gives the best approximation as $f(x) \doteq 0.40579 + 0.33304x - 0.05852x^2 + 0.01346x^3$, with an L^3 error of about 0.00037. The results given by using the tables in conjunction with the principal approximate methods outlined above are as follows.

Collocation: $f(x) \doteq 0.40578 + 0.33325x - 0.05850x^2 + 0.01313x^3$, L^3 error about 0.00039.

Economization with $Q = 5$: $f(x) \doteq 0.40576 + 0.33312x - 0.05833x^2 + 0.01328x^3$, L^3 error about 0.00039.

University of London
King's College
Mathematics Department
London, W.C. 2
England

1. D. Jackson, *Trans. Amer. Math. Soc.*, v. 22, 1921, pp. 117–128, 320–326.
2. C. Lanczos, *Applied Analysis*, Prentice-Hall, Englewood Cliffs, N. J., 1956. MR **18**, 823.
3. C. W. Clenshaw, "The numerical solution of linear differential equations in Chebyshev series," *Proc. Cambridge Philos. Soc.*, v. 53, 1957, pp. 134–149. MR **18**, 516.

* If $a(x)$ is an approximation to a function $f(x)$, we call $L^p(a - f)$ the error in the L^p sense, or simply the L^p error.