A Starting Method for Solving Nonlinear Volterra Integral Equations

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Abstract. In this paper a fifth order starting method is given for Volterra equations of the form $y(t) = f(t) + \int_{x_0}^{t} k(t, s, y(s))\, ds$. Computational examples are given for the method as a starting method for the Gregory-Newton method.

1. Introduction. In this paper we shall consider an $O(h^5)$ starting method for the numerical solution of the nonlinear Volterra integral equation

$$y(t) = f(t) + \int_{x_0}^{t} k(t, s, y(s))\, ds. \tag{1}$$

After stating our algorithm we shall discuss its deriviation and consider some computational examples. In our computational examples we shall consider our method as a starting method for the Gregory-Newton method. The Gregory-Newton method in this context has been discussed by Fox and Goodwin [2], Noble [8], and Todd [11].

2. The Algorithm. The self-starting method described here advances the solution from x_0 to $x_0 + h$, $x_0 + h$ to $x_0 + 2h$, $\cdots$, $x_0 + 5h$ to $x_0 + 6h$. To advance from x_0 to $x_0 + h$ we compute

$$\hat{y}_{1/3} = f\left(x_0 + \frac{h}{3}\right) + \frac{h}{3} k\left(x_0 + \frac{h}{3}, x_0, y_0\right), \tag{2}$$

$$y_{1/3} = f\left(x_0 + \frac{h}{3}\right) + \frac{h}{6}\left[k\left(x_0 + \frac{h}{3}, x_0, y_0\right) + k\left(x_0 + \frac{h}{3}, x_0 + \frac{h}{3}, \hat{y}_{1/3}\right)\right], \tag{3}$$

$$\hat{y}_{2/3} = f\left(x_0 + \frac{2h}{3}\right) + \frac{2h}{3} k\left(x_0 + \frac{2h}{3}, x_0 + \frac{h}{3}, y_{1/3}\right), \tag{4}$$

$$\hat{y}_{1/2} = f\left(x_0 + \frac{h}{2}\right) + \frac{h}{8}\left[k\left(x_0 + \frac{h}{2}, x_0, y_0\right) + 3k\left(x_0 + \frac{h}{2}, x_0 + \frac{h}{3}, y_{1/3}\right)\right], \tag{5}$$

$$\hat{y}_1 = f(x_0 + h) + \frac{h}{4}\left[k(x_0 + h, x_0, y_0) + 3k\left(x_0 + h, x_0 + \frac{2h}{3}, \hat{y}_{2/3}\right)\right], \tag{6}$$

$$\begin{aligned} y_1 = f(x_0 + h) + \frac{h}{6}\Big[&k(x_0 + h, x_0, y_0) \\ &+ 4k\left(x_0 + h, x_0 + \frac{h}{2}, \hat{y}_{1/2}\right) + k(x_0 + h, x_0 + h, \hat{y}_1)\Big]. \end{aligned} \tag{7}$$

To advance from $x_0 + h$ to $x_0 + 2h$ we compute

$$\begin{aligned} \hat{y}_{3/2} = f\left(x_0 + \frac{3h}{2}\right) + \frac{3h}{4}\Big[&k\left(x_0 + \frac{3h}{2}, x_0 + \frac{h}{2}, \hat{y}_{1/2}\right) \\ &+ k\left(x_0 + \frac{3h}{2}, x_0 + h, y_1\right)\Big], \end{aligned} \tag{8}$$

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$$y_{3/2} = f\left(x_0 + \frac{3h}{2}\right) + \frac{3h}{16}\left[k\left(x_0 + \frac{3h}{2}, x_0, y_0\right) + 3k\left(x_0 + \frac{3h}{2}, x_0 + \frac{h}{2}, \hat{y}_{1/2}\right)\right. \tag{9}$$
$$\left. + 3k\left(x_0 + \frac{3h}{2}, x_0 + h, y_1\right) + k\left(x_0 + \frac{3h}{2}, x_0 + \frac{3h}{2}, \hat{y}_{3/2}\right)\right],$$

$$\hat{y}_2 = f(x_0 + 2h) + \frac{2h}{3}\left[k\left(x_0 + 2h, x_0 + \frac{h}{2}, \hat{y}_{1/2}\right) \cdot 2\right. \tag{10}$$
$$\left. - k(x_0 + 2h, x_0 + h, y_1) + 2k\left(x_0 + 2h, x_0 + \frac{3h}{2}, y_{3/2}\right)\right],$$

$$y_2 = f(x_0 + 2h) + \frac{h}{6}[k(x_0 + 2h, x_0, y_0) + 4k(x_0 + h2, x_{1/2}, \hat{y}_{1/2}) \tag{11}$$
$$+ 2k(x_2, x_1, y_1) + 4k(x_2, x_{3/2}, y_{3/2}) + k(x_2, x_2, \hat{y}_2)].$$

To advance from $x_0 + 2h$ to $x_0 + 3h$ we compute

$$\hat{y}_{5/2} = f\left(x_0 + \frac{5h}{2}\right) + \frac{5h}{48}\left[11k\left(x_0 + \frac{5h}{2}, x_0 + \frac{h}{2}, \hat{y}_{1/2}\right)\right. \tag{12}$$
$$+ k\left(x_0 + \frac{5h}{2}, x_0 + h, y_1\right) + k\left(x_0 + \frac{5h}{2}, x_0 + \frac{3h}{2}, y_{3/2}\right)$$
$$\left. + 11k\left(x_0 + \frac{5h}{2}, x_0 + 2h, y_2\right)\right],$$

$$y_{5/2} = f\left(x_0 + \frac{5h}{2}\right) + \frac{5h}{576}\left[19k\left(x_0 + \frac{5h}{2}, x_0, y_0\right)\right. \tag{13}$$
$$+ 75k\left(x_0 + \frac{5h}{2}, x_{1/2}, \hat{y}_{1/2}\right) + 50k\left(x_0 + \frac{5h}{2}, x_1, y_1\right)$$
$$\left. + 50k\left(x_0 + \frac{5h}{2}, x_{3/2}, y_{3/2}\right) + 75k(x_{5/2}, x_2, y_2) + 19k(x_{5/2}, x_{5/2}, \hat{y}_{5/2})\right],$$

$$\hat{y}_3 = f(x_0 + 3h) + \frac{3h}{20}[11k(x_3, x_{1/2}, \hat{y}_{1/2}) - 14k(x_3, x_1, y_1) \tag{14}$$
$$+ 26k(x_3, x_{3/2}, y_{3/2}) - 14k(x_3, x_2, y_2) + 11k(x_3, x_{5/2}, y_{5/2})],$$

$$y_3 = f(x_0 + 3h) + \frac{h}{6}[k(x_3, x_0, y_0) + 4k(x_3, x_{1/2}, \hat{y}_{1/2}) + 2k(x_3, x_1, y_1) \tag{15}$$
$$+ 4k(x_3, x_{3/2}, y_{3/2}) + 2k(x_3, x_2, y_2) + 4k(x_3, x_{5/2}, y_{5/2}) + k(x_3, x_3, \hat{y}_3)].$$

To advance from $x_0 + 3h$ to $x_0 + 4h$ we compute

$$\hat{y}_4 = f(x_0 + 4h) + \frac{4h}{3}[2k(x_4, x_1, y_1) - k(x_4, x_2, y_2) + 2k(x_4, x_3, y_3)], \tag{16}$$

$$y_4 = f(x_0 + 4h) + \frac{4h}{90}[7k(x_0 + 4h, x_0, y_0) + 32k(x_0 + 4h, x_0 + h, y_1) \tag{17}$$
$$+ 12k(x_0 + 4h, x_2, y_2) + 32k(x_4, x_3, y_3) + 7k(x_4, x_4, \hat{y}_4)].$$

To advance from $x_0 + 4h$ to $x_0 + 5h$ we compute

$$\hat{y}_5 = f(x_0 + 5h) + \frac{5h}{24}[11k(x_5, x_1, y_1) + k(x_5, x_2, y_2) + k(x_5, x_3, y_3) + 11k(x_5, x_4, y_4)], \tag{18}$$

$$y_5 = f(x_0 + 5h) + \frac{5h}{288}[19k(x_0 + 5h, x_0, y_0) + 75k(x_0 + 5h, x_0 + h, y_1) + 50k(x_5, x_2, y_2) + 50k(x_5, x_3, y_3) + 75k(x_5, x_4, y_4) + 19k(x_5, x_5, \hat{y}_5)]. \tag{19}$$

To advance from $x_0 + 5h$ to $x_0 + 6h$ we compute

$$\hat{y}_6 = f(x_0 + 6h) + \frac{6h}{20}[11k(x_6, x_1, y_1) - 14k(x_6, x_2, y_2) + 26k(x_6, x_3, y_3) - 14k(x_6, x_4, y_4) + 11k(x_6, x_5, y_5)], \tag{20}$$

$$y_6 = f(x_0 + 6h) + \frac{3h}{10}[k(x_6, x_0, y_0) + 5k(x_6, x_1, y_1) + k(x_6, x_2, y_2) + 6k(x_6, x_3, y_3) + k(x_6, x_4, y_4) + 5k(x_6, x_5, y_5) + k(x_6, x_6, \hat{y}_6)]. \tag{21}$$

3. Derivation of Algorithm. We shall sketch the derivation of the algorithm. Many of the ideas for the algorithm will be found in a paper due to Kuntzmann [5].

If we approximate the integral in (1) by Simpson's rule on the interval $[x_0, x_0 + h]$ we obtain

$$y(x_0 + h) = f(x_0 + h) + \frac{h}{6}\Big[k(x_0 + h, x_0, y_0) + 4k\left(x_0 + h, x_0 + \frac{h}{2}, y\left(x_0 + \frac{h}{2}\right)\right) + k(x_0 + h, x_0 + h, y(x_0 + h))\Big] - \frac{h^5}{2880}k^{\mathrm{IV}}(x_0 + h, \xi, y(\xi)). \tag{22}$$

where $x_0 < \xi < x_0 + h$. Here $y(x_0 + h/2)$ and $y(x_0 + h)$ are not known in the right side of (22). If we are to use (22) we must obtain accurate approximate values for $y(x_0 + h/2)$ and $y(x_0 + h)$. We do this in the following manner. First we note that

$$y(x_0 + h) = f(x_0 + h) + \frac{h}{4}\left[k(x_0 + h, x_0, y_0) + 3k\left(x_0 + h, x_0 + \frac{2h}{3}, y\left(x_0 + \frac{2h}{3}\right)\right)\right] + O(h^4) \tag{23}$$

is an $O(h^4)$ approximation to $y(x_0 + h)$. (This is the Radau two-point rule.) However, here we do not know $y(x_0 + 2h/3)$, but if we could obtain it to $O(h^3)$ then we could use (23). Thus, we attempt to attain an $O(h^3)$ approximation to $y(x_0 + 2h/3)$. This is done by using the midpoint rule

$$y\left(x_0 + \frac{2h}{3}\right) = f\left(x_0 + \frac{2h}{3}\right) + \frac{2h}{3}k\left(x_0 + \frac{2h}{3}, x_0 + \frac{h}{3}, y\left(x_0 + \frac{h}{3}\right)\right) + O(h^3). \tag{24}$$

However here we do not know $y(x_0 + h/3)$ to $O(h^2)$. We obtain it to $O(h^3)$ by using the trapezoidal rule and Taylor's series

$$
\begin{aligned}
y\left(x_0+\frac{h}{3}\right) &= f\left(x_0+\frac{h}{3}\right)+\frac{h}{6}\left[k\left(x_0+\frac{h}{3}, x_0, y_0\right)\right. \\
&\quad \left. + k\left(x_0+\frac{h}{3}, x_0+\frac{h}{3}, y\left(x_0+\frac{h}{3}\right)\right)\right]+O(h^3),
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
y\left(x_0+\frac{h}{3}\right) &= f\left(x_0+\frac{h}{3}\right)+\int_{x_0}^{x_0+h/3}\left[k\left(x_0+\frac{h}{3}, x_0, y_0\right)+O(h)\right] ds \\
&= f\left(x_0+\frac{h}{3}\right)+\frac{h}{3} k\left(x_0+\frac{h}{3}, x_0, y_0\right)+O(h^2).
\end{aligned} \tag{26}
$$

Summarizing the above procedure, we have that formula (23) is used to predict a value for y_1 (Eq. (6)) which is then corrected with (25) (Eq. (7)). Formula (26) is used to predict a value for $y_{1/3}$ (Eq. (2)) which is corrected with (25) (Eq. (3)).

The value of $\hat{y}_{1/2}$ is obtained by approximating the integral in

$$
y\left(x_0+\frac{h}{2}\right)=f\left(x_0+\frac{h}{2}\right)+\int_{x_0}^{x_0+h/2} k(t, s, y(s))\, ds, \qquad t=x_0+\frac{h}{2},
$$

by the Radau two-point rule, disregarding the truncation error and substituting $y_{1/3}$ in for $y(x_0 + h/3)$.

In advancing from $x_0 + h$ to $x_0 + 2h$, we first let x equal to $x_0 + 2h$ in (1) to obtain

$$
y(x_0+2h)=f(x_0+2h)+\int_{x_0}^{x_0+2h} k(x_0+2h, s, y(s))\, ds. \tag{27}
$$

This integral could be evaluated by Simpson's rule if we knew accurate approximate values for $y_{3/2}$ and y_2. We obtain approximate values for $y_{3/2}$ by first using the open Newton-Cotes formula

$$
\begin{aligned}
\hat{y}\left(x_0+\frac{3h}{2}\right) &= f\left(x_0+\frac{3h}{2}\right)+\frac{3h}{4}\left[k\left(x_0+\frac{3h}{2}, x_0+\frac{h}{2}, y_{1/2}\right)\right. \\
&\quad \left. + k\left(x_0+\frac{3h}{2}, x_0+h, y_1\right)\right]+O(h^3)
\end{aligned} \tag{28}
$$

and substituting this value into Simpson's three-eighths' rule on $[x_0, x_0 + 3h/2]$

$$
\begin{aligned}
y\left(x_0+\frac{3h}{2}\right) &= f\left(x_0+\frac{3h}{2}\right)+\frac{3h}{16}\left[k(x_{3/2}, x_0, y_0)+3k(x_{3/2}, x_{1/2}, y_{1/2})\right. \\
&\quad \left. + 3k(x_{3/2}, x_1, y_1)+k(x_{3/2}, x_{3/2}, \hat{y}_{3/2})\right]+O(h^4).
\end{aligned}
$$

An accurate value for $y(x_0 + 2h)$ is obtained by using the Newton-Cotes open formula

$$
\begin{aligned}
y(x_0+2h) &= f(x_0+2h)+\frac{2h}{3}\left[2k(x_2, x_{1/2}, y_{1/2})-k(x_0+2h, x_1, y_1)\right. \\
&\quad \left. + 2k(x_2, x_{3/2}, y_{3/2})\right]+O(h^5).
\end{aligned}
$$

and substituting this result into Simpson's rule

$$y(x_0 + 2h) = f(x_0 + 2h) + \frac{h}{6}[k(x_2, x_0, y_0) + 4k(x_2, x_{1/2}, y_{1/2}) + 2k(x_2, x_1, y_1) + 4k(x_2, x_{3/2}, y_{3/2}) + k(x_2, x_2, y_2)] + O(h^5).$$

To advance from $x_0 + 2h$ to $x_0 + 3h$ we could again use Simpson's rule if we knew accurate approximate values for $y_{5/2}$ and y_3. We proceed as follows. Use the open Newton-Cotes formula

$$\hat{y}_{5/2} = f\left(x_0 + \frac{5h}{2}\right) + \frac{5h}{48}\left[11k\left(x_0 + \frac{5h}{2}, x_{1/2}, y_{1/2}\right) + k\left(x_0 + \frac{5h}{2}, x_1, y_1\right) + k\left(x_0 + \frac{5h}{2}, x_{3/2}, y_{3/2}\right) + 11k(x_{5/2}, x_2, y_2)\right] + O(h^5)$$

along with the closed Newton-Cotes formula

$$y_{5/2} = f\left(x_0 + \frac{5h}{2}\right) + \frac{5h}{576}[19k(x_{5/2}, x_0, y_0) + 75k(x_{5/2}, x_{1/2}, y_{1/2}) + 50k(x_{5/2}, x_1, y_1) + 50k(x_{5/2}, x_{3/2}, y_{3/2}) + 75k(x_{5/2}, x_2, y_2) + 19k(x_{5/2}, x_{5/2}, \hat{y}_{5/2})] + O(h^5).$$

To obtain an approximate value for y at x_3 we use the open Newton-Cotes formula

$$y_3 = f(x_0 + 3h) + \frac{3h}{20}[11k(x_0 + 3h, x_{1/2}, y_{1/2}) - 14k(x_3, x_1, y_1) + 26k(x_3, x_{3/2}, y_{3/2}) - 14k(x_3, x_2, y_2) + 11k(x_3, x_{5/2}, y_{5/2})] + O(h^6)$$

together with Simpson's rule

$$y(x_0 + 3h) = f(x_0 + 3h) + \frac{h}{6}[k(x_0 + 3h, x_0, y_0) + 4k(x_3, x_{1/2}, y_{1/2}) + 2k(x_3, x_1, y_1) + 4k(x_3, x_{3/2}, y_{3/2}) + 2k(x_3, x_2, y_2) + 4k(x_3, x_{5/2}, y_{5/2}) + k(x_3, x_3, y_3)] + O(h^5).$$

It should be noted that the predictor is of higher order than the corrector here.

To advance from $x_0 + 3h$ to $x_0 + 4h$ we approximate the integral in

$$y(x_0 + 4h) = f(x_0 + 4h) + \int_{x_0}^{x_0+4h} k(x_0 + 4h, s, y(s))\, ds$$

by the Newton-Cotes formula

$$y(x_0 + 4h) = f(x_0 + 4h) + \frac{4h}{90}[7k(x_4, x_0, y_0) + 32k(x_4, x_1, y_1) + 12k(x_4, x_2, y_2) + 32k(x_4, x_3, y_3) + 7k(x_4, x_4, y_4)] + O(h^7).$$

Here y_4 is obtained from the open Newton-Cotes formula

$$y(x_0 + 4h) = f(x_0 + 4h) + \frac{4h}{3}[2k(x_4, x_1, y_1) - k(x_4, x_2, y_2) + 2k(x_4, x_3, y_3)] + O(h^5).$$

An approximate value of y at $x_0 + 5h$ is obtained by the open Newton-Cotes formula

$$y(x_0 + 5h) = f(x_0 + 5h) + \frac{5h}{24}[11k(x_5, x_1, y_1) + k(x_5, x_2, y_2) + k(x_5, x_3, y_3) + 11k(x_5, x_4, y_4)] + O(h^5)$$

combined with the closed Newton-Cotes formulae

$$y(x_0 + 5h) = f(x_0 + 5h) + \frac{5h}{288}[19k(x_5, x_0, y_0) + 75k(x_5, x_1, y_1) + 50k(x_5, x_2, y_2) + 50k(x_5, x_3, y_3) + 75k(x_5, x_4, y_4) + 19k(x_5, x_5, y_5)] + O(h^6).$$

To advance from $x_0 + 5h$ to $x_0 + 6h$ we use the open Newton-Cotes formula

$$y_6 = f(x_0 + 6h) + \frac{6h}{20}[11k(x_0 + 6h, x_1, y_1) - 14k(x_6, x_2, y_2) + 26k(x_6, x_3, y_3) - 14k(x_6, x_4, y_4) + 11k(x_6, x_5, y_5)] + O(h^7)$$

together with Weddle's rule

$$y_6 = f(x_0 + 6h) + \frac{3h}{10}[k(x_6, x_0, y_0) + 5k(x_6, x_1, y_1) + k(x_6, x_2, y_2) + 6k(x_6, x_3, y_3) + k(x_6, x_4, y_4) + 5k(x_6, x_5, y_5) + k(x_6, x_6, y_6)] + O(h^7).$$

The Newton-Cotes open and closed formulae and Weddle's rule are given in Milne [7]. For the other integration rules used here, see Hildebrand [3]. It should be noted that we have assumed that the eighth partial derivative of k with respect to s and $y(s)$ exist and is bounded in order to apply our method.

The method under consideration can be applied to systems of integral equations.

4. Use of Gregory-Newton Formulae. The Gregory-Newton Formulae (see Todd [11], Hildebrand [3])

$$\int_{x_0}^{x_0+nh} f(p)\,dp = h\left\{\frac{f(x_0)}{2} + f(x_1) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2}\right\} + \frac{h}{12}\{[f(x_1) - f(x_0)] - [f(x_n) - f(x_{n-1})]\} - \frac{h}{24}\{[f(x_2) - 2f(x_1) + f(x_0)] + [f(x_n) - 2f(x_{n-1}) + f(x_{n-2})]\}$$

$$+ \frac{19h}{720} \{[f(x_3) - 3f(x_2) + 3f(x_1) - f(x_0)] - [f(x_n) - 3f(x_{n-1}) + 3f(x_{n-2}) - f(x_{n-3})]\}$$

$$- \frac{3h}{160} \{[f(x_4) - 4f(x_3) + 6f(x_2) - 4f(x_1) + f(x_0)] + [f(x_n) - 4f(x_{n-1}) + 6f(x_{n-2}) - 4f(x_{n-3}) + f(x_{n-4})]\}$$

$$+ \frac{863h}{60480} [\Delta^5 f(x_0) - \nabla^5 f(x_n)] + \cdots$$

was used by Fox and Goodwin [2] in their treatment of linear Volterra integral equations. In this paper we use the Gregory-Newton formulae through fourth differences to advance the solution from $x = x_0 + 6h$ to any $x = x_0 + Nh$.

Since the integral equation is nonlinear, there is a need for a "predictor" to correspond to the role of the Gregory-Newton formula as "corrector." In our work we have used the following scheme. If we are to advance from $x_0 + (2N - 1)h$ to $x_0 + 2Nh$ use Simpson's rule with step size h, from x_0 through $x_0 + 2Nh - 4h$, then use the open Newton-Cotes formulae

$$\int_{x_0}^{x_4} y\,dx = \frac{4h}{3} [2y_1 - y_2 + 2y_3] + O(h^5)$$

on the interval $[x_0 + 2Nh - 4h, x_0 + 2Nh]$. In case $x = x_0 + (2N - 1)h$ we first integrate from x_0 to $x_0 + 3h$ with Simpson's "three-eighths" rule followed by Simpson's rule until we come to $x_0 + (2N - 1)h - 4h$. Then apply the open Cotes formula used above. This predictor has enabled us to use the Gregory-Newton formula with only two iterations. Before using this, an $O(h^2)$ predictor was used. However seven iterations were necessary in this case. Here the iterations were stopped after a certain number of decimal places of accuracy were achieved.

5. Computational Examples. The following computational examples were computed in Fortran (single precision) on the CDC 1604. By error we mean

$$\text{error} = |\text{ true } - \text{ approximate value }|.$$

Example 1. The integral equation

$$y(t) = 1 - t + \int_0^t (te^{x(t-2x)} + e^{-2x^2}) \cdot (y(x))^2\,dx$$

has the solution $y(x) = e^{x^2}$. It has been considered by Laudet and Oules [6]. We find the following errors.

Example 2. The integral equation

$$y(t) = \frac{2t^{3/2}}{3} + \int_0^t (y(x))^{1/2}\,dx$$

was obtained by integrating the differential equation $y' = x^{1/2} + y^{1/2}$, $y(0) = 0$. This differential equation (see Todd [11], Noble [9]) does not possess a Taylor as pansion about the origin. Its solution about the origin can be written in the series

$$y(x) = \frac{2}{3}x^{3/2} + \frac{4}{7}(2/3)^{1/2}x^{7/4} + \frac{1}{7}x^2 + \frac{1}{49}(2/3)^{1/2}x^{9/4} - \frac{2}{1715}x^{5/2} + \cdots$$

we obtain the following values for x at .1, .2, 1.0 with step sizes .1, .05, .025.

These values compare quite favorably with those obtained by Noble using the Runge-Kutta method (see Noble [9]).

Example 3. The integral equation

$$y(t) = \int_0^t \max(x, y)\, dx$$

was obtained from the differential equation $y' = \max(x, y)$, $y(0) = 0$ (see Burkill [1]). The solution of this differential equation is

$$y(x) = x^2/2 \quad \text{for} \quad x \leqq 2, \qquad y(x) = 2e^{(x-2)} \quad \text{for} \quad x > 2.$$

Thus there is a discontinuity in y'' at $x = 2$.

In this example somewhat better results in the region $x \geqq 2$ were obtained by using the Runge-Kutta method.

Example 4. The integral equation

$$y(t) = 2t + 3 + \int_0^t - y(x)(2(t - x) + 3)\, dx$$

discussed by Todd [11]. The equation has the exact solution $y(t) = 4e^{-2t} - e^{-t}$.

In addition to the above examples the writer has computed examples given by Jones [4], Pouzet [10], Fox and Goodwin [2] and others. These numerical examples are available from the writer in an MRC report.

TABLE 1

x	$h = .05$	$h = .1$	$h = .2$
.05	2.91×10^{-11}		
.1	0	2.91×10^{-10}	
.2	0	2.65×10^{-9}	4.94×10^{-8}
.25	2.91×10^{-11}		
.3	2.91×10^{-11}	3.84×10^{-9}	
.5	0	2.35×10^{-8}	
1.00	2.33×10^{-10}	2.40×10^{-8}	7.65×10^{-5}
2.00	1.80×10^{-6}	9.07×10^{-5}	3.51×10^{-3}
2.50	1.15×10^{-4}	5.79×10^{-3}	

TABLE 2

	$h = .1$	$h = .05$	$h = .025$
$x = .1$	.030711	.030838	.030860
$x = .2$	.093425	.093541	.093621
$x = 1$	1.290677	1.291174	1.291354

TABLE 3

x	$h = .05$	$h = .1$	$h = .2$
.1	1.14×10^{-13}	1.14×10^{-13}	
.2	1.36×10^{-12}	4.55×10^{-13}	4.55×10^{-13}
.3	9.09×10^{-31}	0 (Machine)	
.4	3.64×10^{-12}	5.46×10^{-12}	1.82×10^{-12}
.5	0 (Machine)	0 (Machine)	
1.0	0 (Machine)	0 (Machine)	0 (Machine)
1.4	2.91×10^{-11}	1.46×10^{-11}	1.46×10^{-11}
1.6	8.73×10^{-11}	5.82×10^{-11}	5.82×10^{-11}
1.8	5.82×10^{-11}	5.82×10^{-11}	8.73×10^{-11}
2.0	0 (Machine)	0 (Machine)	1.04×10^{-9}
2.1	8.19×10^{-5}	1.76×10^{-3}	
2.2	2.58×10^{-4}	4.70×10^{-4}	7.27×10^{-3}
2.5	3.43×10^{-4}	1.36×10^{-3}	
3.0		2.26×10^{-3}	8.99×10^{-3}

TABLE 4

x	$h = .05$	$h = .1$	$h = .2$
.1	2.41×10^{-6}	2.44×10^{-5}	
.2	2.48×10^{-6}	6.83×10^{-5}	7.36×10^{-4}
.3	1.38×10^{-7}	4.17×10^{-5}	
.4	1.22×10^{-7}	1.33×10^{-4}	1.68×10^{-3}
.5	2.98×10^{-8}	1.87×10^{-4}	
1.0	8.57×10^{-9}	1.65×10^{-5}	6.66×10^{-3}
1.4	1.49×10^{-9}	4.62×10^{-6}	8.83×10^{-4}
1.6	6.43×10^{-9}	2.37×10^{-6}	3.34×10^{-3}
1.8	9.70×10^{-9}	1.14×10^{-6}	4.63×10^{-4}
2.0	1.25×10^{-8}	4.69×10^{-7}	1.35×10^{-3}
2.5	1.35×10^{-8}	8.57×10^{-8}	
3.0		1.20×10^{-7}	3.45×10^{-4}
4.0		3.70×10^{-8}	4.65×10^{-6}
5.0		8.94×10^{-8}	4.43×10^{-5}

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