Confluent Expansions*

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I. Introduction and Summary. It is well known in special functions, see [1], that the confluent hypergeometric function is a limiting form of the Gaussian hypergeometric function, i.e.

$$\lim_{b\to\infty} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| \frac{z}{b}\right) = {}_1F_1\left(\begin{matrix} a \\ c \end{matrix}\middle| z\right), \tag{1.1}$$

or in series form,

$$\lim_{b\to\infty} \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\,k!}\left(\frac{z}{b}\right)^k = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k k!} z^k, \qquad (\sigma)_\mu \equiv \frac{\Gamma(\sigma+\mu)}{\Gamma(\sigma)}. \tag{1.2}$$

Note that if $a = c$, (1.2) reduces to the familiar limit,

$$\lim_{b\to\infty} (1 - z/b)^{-b} = e^z. \tag{1.3}$$

We will refer to the limit process in (1.2) as a confluence with respect to b. More generally, we will refer to any limit process of the form $\lim_{b\to\infty} \sum_{k=0}^{\infty} f_k(b)$, as a confluence with respect to b, if the functions $f_k(b)$, up to a multiplicative constant dependent on k, are composed of a finite number of multiplicative factors of the form $(\pm b + \omega_1)_k(b + \omega_2)^{-k}$ or their reciprocals, where ω_1 and ω_2 are constants independent of b and k. The value of the limit, if it exists, will be called the confluent limit with respect to b. The reference to b will occasionally be suppressed. As another example of a confluent limit in special functions, see [2], we quote the important classical relation between the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$, and the Bessel functions $J_\alpha(z)$,

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}\left(1 - \frac{z^2}{2n^2}\right) = \left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z), \tag{1.4}$$

or in hypergeometric form,

$$\begin{aligned} &\lim_{n\to\infty} \frac{n^{-\alpha}(n+1)_\alpha}{\Gamma(1+\alpha)}\, {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ 1+\alpha \end{matrix}\middle| \frac{z^2}{4n^2}\right) \\ &\qquad = \frac{1}{\Gamma(1+\alpha)}\, {}_0F_1\left(1+\alpha \middle| -\frac{z^2}{4}\right). \end{aligned} \tag{1.5}$$

In the situation where a confluence with respect to b is possible, it is of interest to consider what happens when b is large but finite. This leads in a natural way to expansions in inverse powers of b or a related variable. Such expansions may be

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either analytic or asymptotic in nature, and will be referred to as analytic or asymptotic confluent expansions respectively, with respect to b. In this paper, several canonical types of confluent expansions will be examined.

For future reference, it is convenient to quote the following Tricomi and Erdélyi result [3].

THEOREM. *If α and β are bounded quantities,*

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim \sum_{j=0}^{\infty} \frac{(-1)^j(\beta-\alpha)_j}{j!} B_j^{(\alpha-\beta+1)}(\alpha)\, z^{\alpha-\beta-j}, \tag{1.6}$$

$$z \to \infty, \quad |\arg(z+\alpha)| < \pi - \delta, \quad \delta > 0; \quad B_0^{(\alpha-\beta+1)}(\alpha) = 1,$$

where the $B_j^{(\alpha-\beta+1)}(\alpha)$ are the generalized Bernoulli polynomials defined by

$$\left(\frac{t}{e^t-1}\right)^{\rho} e^{xt} = \sum_{j=0}^{\infty} \frac{t^j}{j!} B_j^{(\rho)}(x), \qquad |t| < 2\pi. \tag{1.7}$$

We remark that if $\beta - \alpha$ is an integer $\leqq 0$, the asymptotic relation in (1.6) is exact, i.e., asymptotic equality $(\sim)$ can be replaced by ordinary equality $(=)$. Moreover, if $\beta - \alpha$ is an integer > 0, $|z| > \operatorname{Max}\{|\alpha|, |\beta - 1|\}$, then the asymptotic relation in (1.6) is again exact.

II. Analytic Confluent Expansions. In this section we generalize the confluent limits in (1.1) and (1.5). Our results are contained in

THEOREM 1. *Suppose for $|z| < R$,*

$$\sum_{k=0}^{\infty} a_k z^k < \infty; \qquad \sum_{k=0}^{\infty} b_k z^k < \infty. \tag{2.1}$$

Then

$$\begin{aligned} F(z,\sigma) &= \sum_{k=0}^{\infty} a_k \frac{(\sigma)_k}{k!}\left(\frac{z}{\sigma}\right)^k; \\ G(z,\nu,\lambda) &= \sum_{k=0}^{\infty} b_k \frac{(-\nu)_k(\nu+\lambda)_k}{k!\,k!}\left(-\frac{z}{\nu(\nu+\lambda)}\right)^k, \end{aligned} \tag{2.2}$$

converge for $|z| < |\sigma|R$; $|z| < |\nu(\nu+\lambda)|R$, and can be rearranged in descending powers of σ; $\nu(\nu+\lambda)$, to yield the analytic confluent expansions,

$$F(z,\sigma) = \sum_{j=0}^{\infty} g_j(z)\sigma^{-j}, \qquad |z/\sigma| < R; \tag{2.3}$$

$$G(z,\nu,\lambda) = \sum_{j=0}^{\infty} h_j(z,\lambda)[-\nu(\nu+\lambda)]^{-j}, \qquad \left|\frac{z}{\nu(\nu+\lambda)}\right| < R, \tag{2.4}$$

in which the $g_j(z)$ are entire functions of z given explicitly by (2.7), and the $h_j(z,\lambda)$ are polynomials in λ of degree j, whose coefficients are entire functions of z, which are given implicitly by (2.10) and (2.13). For $j \geqq 1$, $g_j(z)$ and $h_j(z,\lambda)$ can be expressed in terms of the derivatives of $g_0(z)$ and $h_0(z,\lambda)$, respectively.

Proof. From the ratio test and (2.1), it follows that $F(z,\sigma)$; $G(z,\nu,\lambda)$ *converge* for $|z| < |\sigma|R$; $|z| < |\nu(\nu+\lambda)|R$. First we prove (2.3). It follows from (1.6) with $z = \sigma$, $\beta = 0$, and $\alpha = k$, together with certain generalized Bernoulli relationships in [4], that

$$\frac{(\sigma)_k}{\sigma^k} = \sum_{j=0}^{k} \frac{(1-k)_j}{j!} B_j^{(k)}(0)\sigma^{-j}, \qquad k = 0, 1, 2, \cdots. \tag{2.5}$$

Clearly the coefficient of σ^{-j} on the right of (2.5) is $\geqq 0$, and $F(z, \sigma)$ is majorized by the series

$$\sum_{k=0}^{\infty} |a_k| \frac{(|\sigma|)_k}{k!} \left|\frac{z}{\sigma}\right|^k, \tag{2.6}$$

which converges for $|z| < |\sigma| R$. Thus (2.5) can be substituted into the series definition of $F(z, \sigma)$ and the resulting series rearranged in powers of σ^{-1}. This leads to (2.3) with

$$g_j(z) = \sum_{k=0}^{\infty} a_k \frac{(1-k)_j B_j^{(k)}(0)}{j!\,k!} z^k. \tag{2.7}$$

To express $g_j(z)$ in terms of the derivatives of $g_0(z)$, we merely note that for fixed j, $(1-k)_j B_j^{(k)}(0)$ is a polynomial in k of degree $2j$, and that it can be written in factorial powers of k, e.g. if $j = 3$,

$$\begin{aligned} &(1-k)_3 B_3^{(k)}(0) \\ &\quad = \frac{k(k-1)(k-2)(k-3)}{8}[(k-4)(k-5) + 8(k-4) + 12]. \end{aligned} \tag{2.8}$$

Substitution of this sort of factorization into (2.7) leads immediately to an expression for $g_j(z)$ in terms of the derivatives of $g_0(z)$. The first few of the $g_j(z)$ are

$$\begin{aligned} g(z) &= g_0(z) = \sum_{k=0}^{\infty} (a_k/k!) z^k, \qquad g_1(z) = (z^2/2) g^{(2)}(z), \\ g_2(z) &= (z^3/3) g^{(3)}(z) + (z^4/8) g^{(4)}(z), \\ g_3(z) &= (z^4/4) g^{(4)}(z) + (z^5/6) g^{(5)}(z). \end{aligned} \tag{2.9}$$

We now prove (2.4). For $\nu(\nu + \lambda) \neq 0$, k and j integers $\geqq 0$, define the polynomials $C_{j,k}(\lambda)$ by

$$\begin{aligned} &\frac{(-1)^k(-\nu)_k(\nu+\lambda)_k}{[\nu(\nu+\lambda)]^k} \\ &\quad = (1 - k/\nu)^{-1} \prod_{j=0}^{k} (1 - j/\nu) \cdot (1 + k/(\nu+\lambda))^{-1} \prod_{j=0}^{k} (1 + j/(\nu+\lambda)), \\ &\quad = \left(1 + \frac{k(k+\lambda)}{(-\nu)(\nu+\lambda)}\right)^{-1} \prod_{j=0}^{k} \left(1 + \frac{j(j+\lambda)}{(-\nu)(\nu+\lambda)}\right), \\ &\quad = \sum_{j=0}^{k} C_{j,k}(\lambda)[(-\nu)(\nu+\lambda)]^{-j}. \end{aligned} \tag{2.10}$$

From (2.10), it is easy to see that $C_{j,k}(\lambda)$ is a polynomial in λ of order j, whose coefficients are positive. Thus $G(z, \nu, \lambda)$ is majorized by the series

$$\sum_{k=0}^{\infty} \frac{|b_k||z|^k}{k!k!} \sum_{j=0}^{k} C_{j,k}(|\lambda|)\,|\nu(\nu+\lambda)|^{-j} = |b_0| + \sum_{k=1}^{\infty} \frac{|b_k||z|^k}{k!k!} \prod_{j=0}^{k-1}\left(1 + \frac{j(j+|\lambda|)}{|\nu(\nu+\lambda)|}\right), \tag{2.11}$$

which in turn is majorized by

$$|b_0| + \sum_{k=1}^{\infty} \frac{|b_k||z|^k}{k!k!} \prod_{j=0}^{k-1}\left(\frac{(j+|\nu|)(j+|\nu|+|\lambda|)}{|\nu(\nu+\lambda)|}\right) = \sum_{k=0}^{\infty} |b_k| \frac{(|\nu|)_k(|\nu|+|\lambda|)_k}{k!k!}\left|\frac{z}{\nu(\nu+\lambda)}\right|^k, \quad |z| < |\nu(\nu+\lambda)|R. \tag{2.12}$$

Thus $G(z, \nu, \lambda)$ can be rearranged in descending powers of $\nu(\nu + \lambda)$, and since ν and λ were arbitrary,

$$h_j(z, \lambda) = \sum_{k=0}^{\infty} \frac{b_k C_{j,k}(\lambda)}{k!k!} z^k, \tag{2.13}$$

is a polynomial in λ of order j, which converges for arbitrary z. The final statement of the theorem follows as before. The $C_{j,k}(\lambda)$ can be defined recursively. Multiplying (2.10) through by $(\nu - k)(\nu + \lambda + k)[\nu(\nu + \lambda)]^{-1}$, one is led to the relation

$$C_{j,k+1}(\lambda) - C_{j,k}(\lambda) = k(k+\lambda)C_{j-1,k}(\lambda); \qquad k, j \geqq 0. \tag{2.14}$$

From (2.10) and (2.14), it follows that

$$C_{j+1,k}(\lambda) = \sum_{m=0}^{k-1} m(m+\lambda)C_{j,m}(\lambda), \qquad C_{0,m}(\lambda) = 1, \qquad m \geqq 0. \tag{2.15}$$

Incorporating the same type of factorization as used to write $g_j(z)$ in terms of the derivatives of $g_0(z)$, we have by explicit computation from (2.15) for the first few $C_{j\,k}(\lambda)$,

$$\begin{aligned} C_{0,k}(\lambda) &= 1, \\ C_{1,k}(\lambda) &= \frac{k(k-1)}{6}[2(k-2)+3] + \frac{k(k-1)}{2}\lambda, \\ C_{2,k}(\lambda) &= \frac{k(k-1)(k-2)}{360}[20(k-3)(k-4)(k-5) \\ &\qquad + 204(k-3)(k-4) + 495(k-3) + 240] \\ &\quad + \frac{k(k-1)(k-2)}{6}[(k-3)(k-4) + 6(k-4) + 6]\lambda \\ &\quad + \frac{k(k-1)(k-2)}{24}[3(k-3)+8]\lambda^2. \end{aligned} \tag{2.16}$$

Substitution of (2.16) into (2.13) then yields for the first few $h_j(z, \lambda)$,

$$h(z) = h_0(z, \lambda) = \sum_{k=0}^{\infty} (b_k/k!\,k!)z^k,$$

$$(2.17)\quad h_1(z, \lambda) = ((\lambda + 1)/2)z^2h^{(2)}(z) + (z^3/3)h^{(3)}(z),$$

$$h_2(z, \lambda) = ((\lambda^2 + 3\lambda + 2)/3)z^3h^{(3)}(z) + ((\lambda^2 + 8\lambda + 11)/8)z^4h^{(4)}(z) + ((5\lambda + 17)/30)z^5h^{(5)}(z) + (z^6/18)h^{(6)}(z).$$

This completes the proof of the theorem.

Remark I. The coefficients of the $g^{(j)}(z)$ in (2.9) are independent of the identity of the function $g_0(z)$, and thus can be deduced from the special case when $g_0(z) = e^z$, i.e.

$$(2.18)\qquad F(z, \sigma) = (1 - z/\sigma)^{-\sigma} = e^z \exp\left\{\sum_{j=2}^{\infty} \frac{\sigma}{j}\left(\frac{z}{\sigma}\right)^j\right\}.$$

Similar remarks apply to the $h^{(j)}(z)$ in (2.17).

Remark II. The characterization of $g_j(z)$; $h_j(z, \lambda)$, given in (2.9); (2.17), is particularly convenient when working with generalized hypergeometric functions, since for m, p, q, integers $\geqq 0$, $p \leqq q + 1$,

$$(2.19)\qquad \frac{d^m}{dz^m}\,{}_pF_q\left(\begin{matrix}\alpha_1, \cdots, \alpha_p\\ \beta_1, \cdots, \beta_q\end{matrix}\middle| z\right) = \left(\prod_{j=1}^{p} (\alpha_j)_m \Big/ \prod_{j=1}^{q} (\beta_j)_m\right) {}_pF_q\left(\begin{matrix}m + \alpha_1, \cdots, m + \alpha_p\\ m + \beta_1, \cdots, m + \beta_q\end{matrix}\middle| z\right).$$

Remark III. If $b_k = k!\,a_k$, the functions F and G are related by a confluence limit, i.e.

$$(2.20)\qquad \lim_{\lambda\to\infty} G(z, \nu, \lambda) = F(z, -\nu).$$

Remark IV. If $-\sigma = \nu = n$, an integer $\geqq 0$, then $F(z, -n)$ and $G(z, n, \lambda)$ are polynomials in z of degree n. Moreover, if the hypothesis (2.1) of Theorem 1 is replaced by the weaker hypothesis that for $|z| < R^*$,

$$(2.21)\qquad \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k < \infty; \qquad \sum_{k=0}^{\infty} \frac{b_k}{k!\,k!} z^k < \infty,$$

then the $g_j(z)$; $h_j(z, \lambda)$, which are now only defined for $|z| < R^*$, are the Poincaré coefficients of $F(z, -n)$; $G(z, n, \lambda)$ as $n \to \infty$. As is indicated by Remark III and the proof of Theorem 1, the proof of this fact for $F(z - n)$ is similar to, but simpler than, the proof of the corresponding fact for $G(z, n, \lambda)$. The result for $G(z, n, \lambda)$ follows readily from the following

LEMMA 1. *Suppose m an integer $\geqq 0$, and k, n integers such that $1 \leqq k \leqq n$. Then for n sufficiently large, the $C_{j,k}(\lambda)$ defined by (2.10) satisfy the inequality,*

$$(2.22)\qquad (m!)\left|\frac{(-1)^k(-n)_k(n + \lambda)_k}{[n(n + \lambda)]^k} - \sum_{j=0}^{m-1} C_{j,k}(\lambda)[-n(n + \lambda)]^{-j}\right| \cdot |n(n + \lambda)|^m k^{-2m}(k + |\lambda|)^{-m} \leqq e^{\mu(n+\alpha)},$$

$$\lambda = \alpha + i\beta; \qquad \alpha, \beta \text{ real}; \qquad \mu = \text{Max}\,\{0, -\alpha(3|\alpha| + 2|\beta|)/4\}.$$

To prove (2.22), it is sufficient to notice that for n sufficiently large,

$$\max_{0 \leqq u \leqq 1; 0 \leqq t \leqq n} \left| 1 - \frac{ut(t+\lambda)}{n(n+\lambda)} \right| \leqq 1 + \frac{\mu}{n(n+\alpha)}, \tag{2.23}$$

and hence by direct computation, that

$$\begin{aligned} &\max_{0 \leqq u \leqq 1} | H_k^{(m)}(-u[n(n+\lambda)]^{-1} | \\ &\leqq k^{2m}(k + |\lambda|)^m(1 + \mu[n(n+\alpha)]^{-1})^n \leqq k^{2m}(k+|\lambda|)^m e^{\mu/(n+\alpha)}, \\ &H_k(x) = \prod_{j=0}^{k-1} [1 + j(j+\lambda)x]. \end{aligned} \tag{2.24}$$

Combining (2.24) with Taylor's theorem, one arrives at (2.22).

III. Asymptotic Confluent Expansions. Here we give two canonical examples of asymptotic confluent expansions. Our first result is contained in

THEOREM 2. *Suppose for* $|z| < R$,

$$\sum_{k=0}^{\infty} c_k z^k < \infty. \tag{3.1}$$

Then

$$T(z, \rho) = \sum_{k=0}^{\infty} \frac{c_k}{(\rho)_k} (\rho z)^k, \tag{3.2}$$

converges for all z, $\rho \neq 0, -1, \cdots$, *and possesses the asymptotic confluent expansion*

$$\begin{aligned} &T(z, \rho) \sim \sum_{j=0}^{\infty} f_j(z)(-\rho)^{-j}, \\ &\rho \to \infty, \quad |\arg \rho| \leqq \pi - \delta, \quad 0 < \delta \leqq \pi/2, \quad |zr_\rho| \leqq R; \\ &r_\rho = 1 \quad \textit{if} \quad |\arg \rho| \leqq \pi/2, \\ &\quad = |\sin(\arg \rho)|^{-1} \quad \text{if} \quad \pi/2 \leqq |\arg \rho| \leqq \pi - \delta, \end{aligned} \tag{3.3}$$

in which the $f_j(z)$ *are functions analytic in* $|z| < R$ *given explicitly by* (3.8). *Moreover, for* $j \geqq 1$, $f_j(z)$ *can be expressed in terms of the derivatives of* $f_0(z)$.

Proof. From the ratio test, it follows that $T(z, \rho)$ converges under the stated conditions. Next we note that (1.6) with $z = \rho$, $\alpha = 0$, $\beta = k$, reduces to

$$\frac{\rho^k}{(\rho)_k} = \sum_{j=0}^{\infty} \beta_{j,k} \rho^{-j}, \qquad \beta_{j,k} = \frac{(-1)^j (k)_j}{j!} B_j^{(1-k)}(0). \tag{3.4}$$

Before computing the Poincaré coefficients of $T(z, \rho)$, we need the following

LEMMA 2. *Suppose n an integer* $\geqq 0$, *and k an integer* $\geqq 1$. *Then for ρ sufficiently large, the* $\beta_{j,k}$ *defined by* (3.4) *satisfy the inequality,*

$$\begin{aligned} &\left| \frac{\rho^k}{(\rho)_k} - \sum_{j=0}^{n-1} \beta_{j,k} \rho^{-j} \right| |\rho|^n (r_\rho)^{1-n-k} k^{-2n} \leqq 1, \\ &|\arg \rho| \leqq \pi - \delta, \qquad 0 < \delta \leqq \pi/2. \end{aligned} \tag{3.5}$$

Proof. If $|\arg \rho| \leqq \pi - \delta$ and $ju \geqq 0$, then

$$|1 + ju\rho^{-1}|^{-1} \leqq r_\rho . \tag{3.6}$$

It follows by an induction proof on n, that

$$\begin{aligned} &\operatorname*{Max}_{0 \leqq u \leqq 1} |F_k^{(n)}(u\rho^{-1})| \leqq (n!)(r_\rho)^{k+n-1}k^{2n}, \qquad n = 0, 1, \cdots, \\ &F_k(x) = \prod_{j=0}^{k-1} (1 + jx)^{-1}, \qquad k = 1, 2, \cdots . \end{aligned} \tag{3.7}$$

Eq. (3.7) combined with Taylor's theorem, yields (3.5) and completes the lemma.

Now set

$$f_j(z) = \sum_{k=0}^{\infty} c_k(-1)^j \beta_{j,k} z^k = \sum_{k=0}^{\infty} c_k \frac{(k)_j B_j^{(1-k)}(0)}{j!} z^k, \qquad j = 0, 1, \cdots . \tag{3.8}$$

Clearly, for fixed j, the coefficient of $c_k z^k$ in (3.8) is a polynomial in k of degree $(2j)$. Thus the functions $f_j(z)$ are analytic in $|z| < R$. We now show that the $(-1)^j f_j(z)$ are the Poincaré coefficients of $T(z, \rho)$ at $\rho = \infty$, i.e. for arg ρ fixed, $|\arg \rho| \leqq \pi - \delta$,

$$\begin{aligned} &\lim_{\rho \to \infty} (-\rho)^n \left[T(z, \rho) - \sum_{j=0}^{n-1} f_j(z)(-\rho)^{-j} \right] \\ &\qquad = \frac{c_0}{\Gamma(1 - n)} + \sum_{k=1}^{\infty} c_k z^k \lim_{\rho \to \infty} (-\rho)^n \left\{ \frac{\rho^k}{(\rho)_k} - \sum_{j=0}^{n-1} \beta_{j,k} \rho^{-j} \right\}, \\ &\qquad = \frac{c_0}{\Gamma(1 - n)} + \sum_{k=1}^{\infty} c_k \beta_{n,k} (-1)^n z^k \\ &\qquad = f_n(z). \end{aligned} \tag{3.9}$$

The interchange of limit processes in (3.9) follows from the fact that, in view of the lemma, the original series in (3.9) is majorized by the series

$$\frac{|c_0|}{\Gamma(1 - n)} + (r_\rho)^n \sum_{k=1}^{\infty} k^{2n} |c_k| |zr_\rho|^{k-1}, \tag{3.10}$$

which converges for $|zr_\rho| < R$. Note that since r_ρ is a function of arg ρ only, the convergence is uniform in ρ on the ray $t \exp(i \arg \rho)$, $|\rho| \leqq t \leqq \infty$, $|\arg \rho| \leqq \pi - \delta$. Finally, the representation of $f_j(z)$ in terms of the derivatives of $f_0(z)$ follows as in Theorem 1, and the first few are,

$$\begin{aligned} &f(z) = f_0(z) = \sum_{k=0}^{\infty} c_k z^k, \qquad f_1(z) = (z^2/2) f^{(2)}(z), \\ &f_2(z) = (z^2/2) f^{(2)}(z) + (2z^3/3) f^{(3)}(z) + (z^4/8) f^{(4)}(z), \end{aligned} \tag{3.11}$$

which completes the proof of the theorem.

As a final canonical example of a confluent situation, we prove the following

THEOREM 3. *Suppose for* $|z| < \mathrm{R}$,

$$v(z) = \sum_{k=0}^{\infty} d_k z^k < \infty . \tag{3.12}$$

Then

$$S(z, \sigma, a, b) = \sum_{k=0}^{\infty} d_k \frac{(\sigma + a)_k}{(\sigma + b)_k} z^k, \tag{3.13}$$

converges for $|z| < R$, $\sigma + b \neq 0, -1, \cdots$, *and can be rearranged in the region* $|z| < R/2$, *to yield the expansion*

$$S(z, \sigma, a, b) = \sum_{j=0}^{\infty} \frac{(b - a)_j(-z)^j}{(\sigma + b)_j j!} v^{(j)}(z). \tag{3.14}$$

If a *and* b *are bounded quantities* (3.14) *holds asymptotically in the larger region* $|z| < R$, *i.e., if* n *is an integer* > 0,

$$\begin{gathered} S(z, \sigma\, a, b) = \sum_{j=0}^{n-1} \frac{(b - a)_j(-z)^j}{(\sigma + b)_j\, j!} v^{(j)}(z) + O(\sigma^{-n}), \\ \sigma \to \infty, \qquad |\arg(\sigma + b)| \leqq \pi - \delta, \qquad \delta > 0, \qquad |z| < R. \end{gathered} \tag{3.15}$$

If $[(\sigma + b)_j]^{-1}$ *is expanded in powers of* σ^{-1} (3.15) *can be written as an asymptotic confluent expansion in* σ^{-1}.

Proof. From the ratio test, it follows that $S(z, \sigma, a, b)$ converges under the stated conditions. One sees from Gauss's formula for a ${}_2F_1$ of unit argument, [1], that if $\sigma + b \neq 0, -1, \cdots$,

$$\sum_{j=0}^{k} \frac{(-k)_j(b - a)_j}{(\sigma + b)_j\, j!} = {}_2F_1\left(\begin{matrix}-k, b - a \\ \sigma + b\end{matrix}\middle| 1\right) = \frac{(\sigma + a)_k}{(\sigma + b)_k}, \quad k = 0, 1, \cdots. \tag{3.16}$$

Assume that $|z| < z_0 < R$. Then $d_k = O(z_0^{-k})$ uniformly in k, as $k \to \infty$. Thus the right-hand side of (3.14) up to a multiplicative constant is majorized by

$$\begin{aligned} &\sum_{j=0}^{\infty} \left|\frac{(b - a)_j}{(\sigma + b)_j}\right| \frac{1}{j!} \sum_{k=0}^{\infty} (-1)^j(-k)_j \left|\frac{z}{z_0}\right|^k \\ &\qquad = (1 - z/z_0)^{-1} \sum_{j=0}^{\infty} \left|\frac{(b - a)_j}{(\sigma + b)_j}\right| \left(\left|\frac{z_0}{z}\right| - 1\right)^{-j}, \end{aligned} \tag{3.17}$$

which converges for $|z| < |z_0|/2$. Thus the right hand side of (3.14) can be arranged in powers of z, establishing (3.14). To prove the asymptotic expansion (3.15), we merely remark that the same methods used in Theorem 3 can be used to establish the existence of a Poincaré asymptotic expansion of $S(z, \sigma, a, b)$ in powers of σ^{-1}, under the stated conditions of the theorem. In the common region $|z| < R/2$, both the asymptotic expansion in σ^{-1} and (3.15) must agree when $[(\sigma + b)_j]^{-1}$ is expanded in powers of σ^{-1}. This is sufficient to identify the Poincaré coefficients, and establish (3.15). This completes the proof of the theorem.

Remark V. The region $|z| < R/2$ is, in general, the largest circular region in which the right-hand side of (3.14) can converge. This follows from the special case of Theorem 3, known as Euler's formula, see [1],

$$\begin{aligned} {}_2F_1\left(\begin{matrix}\sigma + a, \alpha \\ \sigma + b\end{matrix}\middle| z\right) &= \sum_{j=0}^{\infty} \frac{(b - a)_j(-z)^j}{(\sigma + b)_j\, j!} \frac{d^j}{dz^j} \{(1 - z)^{-\alpha}\} \\ &= (1 - z)^{-\alpha}\, {}_2F_1\left(\begin{matrix}b - a, \alpha \\ \sigma + b\end{matrix}\middle| \frac{z}{z + 1}\right). \end{aligned} \tag{3.18}$$

In this example, $R = 1$, but the right-hand side of (3.18) converges only for $\mathrm{Re}\,(z) < \frac{1}{2}$. Also note in this example that the right-hand side of (3.18) analytically continues the left-hand side of (3.18) outside its original circle of convergence, i.e. the unit circle.

Remark VI. In the special case $S(z, \sigma, a, b)$ is a hypergeometric series, (3.14) yields a proof of the fact that whenever a convergent hypergeometric series has a numerator parameter differing from a denominator parameter by a positive integer m, that the hypergeometric series can be written as the sum of m hypergeometric series of lower order. Although there are many examples of such formulae in the literature, this result seems never to have been proved in general.

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