

Chebyshev Approximations for the Natural Logarithm of the Gamma Function*

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Abstract. Rational Chebyshev approximations are given for the natural logarithm of the real gamma function for arguments in the intervals [0.5, 1.5], [1.5, 4.0] and [4.0, 12.0]. Maximal relative errors range down to 1×10^{-17} .

1. Introduction. The computation of the gamma function for a real positive argument x has been facilitated by the recent work of Rice [1], [2] giving rational Chebyshev approximations over the interval [2, 3]. Rice has also generated approximations for $\ln \Gamma(x)$, $x \geq 12$, of the form

$$(1) \quad \ln \Gamma(x) \cong (x - \frac{1}{2}) \ln x - x + \ln (2\pi)^{1/2} + (1/x)P_n(1/x^2)/Q_m(1/x^2)$$

where $P_n(y)$ and $Q_m(y)$ are polynomials of degree n and m , respectively [2].

The computation of $\ln \Gamma(x)$ can thus be carried out directly from (1) for x sufficiently large. For smaller x it is generally necessary to compute the gamma function first and then to take the natural logarithm. Compared to evaluating a rational function for $\ln \Gamma(x)$ directly, this process is fairly expensive. It is also frequently unstable numerically when x is in the neighborhood of 1 or 2, where the function vanishes.

This paper presents portions of the arrays of best rational approximations for $\ln \Gamma(x)$ on the intervals [.5, 1.5], [1.5, 4.0], and [4.0, 12.0], thus complementing Rice's results. Rice has termed such arrays L_∞ Walsh arrays.

2. The Approximation Forms. The approximation forms and intervals are:

$$R_{n,m}^{0*}(x) = -\ln x + R_{n,m}^{1*}(x + 1), \quad 0 < x \leq 0.5;$$

$$R_{n,m}^{1*}(x) = (x - 1)R_{n,m}^1(x), \quad 0.5 \leq x \leq 1.5;$$

$$R_{n,m}^{2*}(x) = (x - 2)R_{n,m}^2(x), \quad 1.5 \leq x \leq 4.0;$$

and

$$R_{n,m}^{3*}(x) = R_{n,m}^3(x), \quad 4.0 \leq x \leq 12.0;$$

where here and in the following $R_{n,m}^i(x) = P_n^i(x)/Q_m^i(x)$, a ratio of polynomials. These forms correctly contain the logarithmic behaviour of the function as x approaches zero from above, and the zeroes at $x = 1$ and $x = 2$. The particular partitioning of the interval [0.5, 12.0] was chosen because (i) reasonable accuracy is obtained in each subinterval for modest values of n and m , and (ii) the approximations are somewhat balanced, i.e., the maximal errors in each subinterval for a given choice of n, m are approximately the same.

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TABLE II

Coefficients for $R_{n,n}^{1*}(x) = (x - 1) \sum_{i=0}^n P_i^1 x^i / \sum_{i=0}^n Q_i^1 x^i, .5 \leq x \leq 1.5$

i	P_i^1	Q_i^1
$n = 1$		
0	-2.2105 (00)	9.6932 (-01)
1	1.0740 (00)	1.0000 (00)
$n = 2$		
0	-2.60753 8 (00)	8.01323 7 (-01)
1	-2.74306 9 (00)	3.96665 3 (00)
2	2.02119 8 (00)	1.00000 0 (00)
$n = 3$		
0	-2.83354 476 (00)	7.26957 927 (-01)
1	-1.17257 247 (01)	8.00937 488 (00)
2	1.27473 843 (00)	8.69084 166 (00)
3	2.64807 840 (00)	1.00000 000 (00)
$n = 4$		
0	-2.66685 51149 5 (00)	6.07771 38777 1 (-01)
1	-2.44387 53423 7 (01)	1.19400 90572 1 (01)
2	-2.19698 95892 8 (01)	3.14690 11574 9 (01)
3	1.11667 54126 2 (01)	1.52346 87407 0 (01)
4	3.13060 54762 3 (00)	1.00000 00000 0 (00)
$n = 5$		
0	-2.60940 66054 623 (00)	5.45875 04274 950 (-01)
1	-4.15090 18875 434 (01)	1.66749 69701 154 (01)
2	-9.55641 17677 317 (01)	7.85560 98036 754 (01)
3	-1.18114 39967 596 (01)	8.72893 05773 548 (01)
4	2.81137 44347 038 (01)	2.35907 62639 739 (01)
5	3.51735 89912 443 (00)	1.00000 00000 000 (00)
$n = 6$		
0	-2.33590 98949 51284 (00)	4.57174 20282 50299 (-01)
1	-5.75008 93603 04123 (01)	2.01068 51344 33395 (01)
2	-2.45872 61722 29242 (02)	1.50068 39064 89095 (02)
3	-2.15135 13573 72570 (02)	3.08829 54973 42428 (02)
4	5.55840 45723 51531 (01)	1.93877 84034 37713 (02)
5	5.27068 93753 00983 (01)	3.37330 47907 07074 (01)
6	3.84287 36567 45991 (00)	1.00000 00000 00000 (00)
$n = 7$		
0	-2.20884 39972 16182 306 (00)	4.09779 29210 92615 065 (-01)
1	-7.74106 40713 32953 034 (01)	2.44351 96625 06311 704 (01)
2	-5.17638 34980 23217 924 (02)	2.62308 34702 69460 180 (02)
3	-9.22261 37288 01521 582 (02)	8.46075 53620 20782 006 (02)
4	-2.61721 85838 56145 190 (02)	9.51323 59767 97059 772 (02)
5	2.43175 24352 44210 223 (02)	3.77837 24848 23942 081 (02)
6	8.56898 20628 31317 339 (01)	4.56467 71875 85907 957 (01)
7	4.12084 31858 47770 031 (00)	1.00000 00000 00000 000 (00)

TABLE III

Coefficients for $R_{n,n}^{2*}(x) = (x - 2) \sum_{i=0}^n P_i^2 x^i / \sum_{i=0}^n Q_i^2 x^i$, $1.5 \leq x \leq 4.0$

i	P_i^2		Q_i^2	
$n = 1$				
0	-2.0456	(00)	3.2610	(00)
1	2.1327	(00)	1.0000	(00)
$n = 2$				
0	-8.50601 8	(00)	7.88189 3	(00)
1	5.48295 5	(00)	1.13611 2	(01)
2	3.04259 5	(00)	1.00000 0	(00)
$n = 3$				
0	-2.62608 1587	(01)	1.86074 2491	(01)
1	-8.39906 1479	(00)	6.76133 4331	(01)
2	3.09835 8770	(01)	2.47764 2509	(01)
3	3.68073 1176	(00)	1.00000 0000	(00)
$n = 4$				
0	-7.83359 29944 9	(01)	4.70668 76606 0	(01)
1	-1.42046 29668 8	(02)	3.13399 21589 4	(02)
2	1.37519 41641 6	(02)	2.63505 07472 1	(02)
3	7.86994 92415 4	(01)	4.33400 02251 4	(01)
4	4.16438 92222 8	(00)	1.00000 00000 0	(00)
$n = 5$				
0	-2.16192 29262 4703	(02)	1.16412 65946 1333	(02)
1	-8.27790 89780 9598	(02)	1.20459 29366 3292	(03)
2	1.82987 82201 2009	(02)	1.85645 03568 6087	(03)
3	7.06543 70015 4966	(02)	7.05287 06971 5149	(02)
4	1.49903 66270 9861	(02)	6.65573 50746 7416	(01)
5	4.54827 47772 3909	(00)	1.00000 00000 0000	(00)
$n = 6$				
0	-5.60177 73537 80387 7	(02)	2.76785 83623 80410 1	(02)
1	-3.69298 34005 59128 2	(03)	4.16994 15153 20023 1	(03)
2	-1.97780 70769 84164 6	(03)	1.04595 76594 05895 9	(04)
3	3.79751 24011 52511 8	(03)	7.23400 87928 94807 1	(03)
4	2.17973 66058 89591 5	(03)	1.56120 45277 92863 5	(03)
5	2.48845 25168 57407 6	(02)	9.50999 17418 20893 8	(01)
6	4.87402 01396 83863 6	(00)	1.00000 00000 00000 0	(00)
$n = 7$				
0	-1.51383 18341 15066 7785	(03)	6.98327 41405 73510 2159	(02)
1	-1.50863 02287 66725 0272	(04)	1.44020 90371 70085 2304	(04)
2	-2.06482 94205 32528 3281	(04)	5.26228 63838 41199 2470	(04)
3	1.20431 73809 87164 0151	(04)	5.71202 55396 02502 9854	(04)
4	1.95536 05540 63044 9846	(04)	2.20295 62144 15663 6889	(04)
5	5.26898 32559 14981 2458	(03)	3.03990 30414 39439 8824	(03)
6	3.77510 67979 72170 2241	(02)	1.28909 31890 12957 6873	(02)
7	5.15505 76176 40817 1704	(00)	1.00000 00000 00000 0000	(00)

TABLE IV

Coefficients for $R_{n,n}^{3*}(x) = \sum_{i=0}^n P_i^3 x^i / \sum_{i=0}^n Q_i^3 x^i$, $4.0 \leq x \leq 12.0$

i	P_i^3	Q_i^3
$n = 1$		
0	9.910 (01)	-3.122 (01)
1	-3.677 (01)	1.000 (00)
$n = 2$		
0	-1.63774 4 (02)	-8.40784 3 (02)
1	7.74856 9 (02)	-1.26360 0 (02)
2	-3.32436 3 (02)	1.00000 0 (00)
$n = 3$		
0	-1.37219 936 (04)	-1.13338 846 (04)
1	2.05137 737 (04)	-6.27235 378 (03)
2	-4.71572 823 (03)	-3.02297 001 (02)
3	-1.04210 703 (03)	1.00000 000 (00)
$n = 4$		
0	-2.12159 57232 3 (05)	-1.16328 49500 4 (05)
1	2.30661 51061 6 (05)	-1.46025 93751 1 (05)
2	2.74647 64470 5 (04)	-2.42357 40962 9 (04)
3	-4.02621 11997 5 (04)	-5.70691 00932 4 (02)
4	-2.29660 72978 0 (03)	1.00000 00000 0 (00)
$n = 5$		
0	-2.42731 13085 758 (06)	-1.05424 82321 634 (06)
1	1.38608 69828 508 (06)	-2.45157 05199 457 (06)
2	1.85377 73351 564 (06)	-8.62741 86723 037 (05)
3	-6.42799 27530 351 (05)	-6.77712 58633 073 (04)
4	-1.55159 71577 126 (05)	-9.41366 13234 388 (02)
5	-4.21052 09252 847 (03)	1.00000 00000 000 (00)
$n = 6$		
0	-2.48043 69488 28593 (07)	-9.16055 82863 71317 (06)
1	-3.35677 82814 54576 (06)	-3.45441 75093 34395 (07)
2	3.63218 04931 54257 (07)	-2.09696 23255 80444 (07)
3	-2.94234 45930 32234 (06)	-3.41525 17108 01107 (06)
4	-4.75045 94653 43956 (06)	-1.55528 90280 85353 (05)
5	-4.30699 69819 57098 (05)	-1.42168 29839 65146 (03)
6	-6.88062 40094 59425 (03)	1.00000 00000 00000 (00)
$n = 7$		
0	-2.40798 69801 73375 493 (08)	-7.90261 11141 87634 109 (07)
1	-2.44832 17690 32881 564 (08)	-4.35370 71480 43741 914 (08)
2	4.81807 71027 73628 010 (08)	-4.04435 92829 14355 059 (08)
3	1.11938 53542 99855 449 (08)	-1.11925 41162 63318 226 (08)
4	-8.73167 54382 38386 656 (07)	-1.04857 75830 49937 280 (07)
5	-1.97183 01158 60920 573 (07)	-3.11406 28473 40678 552 (05)
6	-9.82710 22814 20492 083 (05)	-2.01527 51955 00482 591 (03)
7	-1.03770 16517 32974 263 (04)	1.00000 00000 00000 000 (00)

the original function routines. In all cases maximal errors agreed, within roundoff, in magnitude and position with those given by the error curves in the Remes algorithm.

4. Results. Table I lists the values of

$$E_{n,m}^{i*} = -100 \log \left\| \frac{\ln \Gamma(x) - R_{n,m}^{i*}(x)}{\ln \Gamma(x)} \right\|_{\infty}$$

for the initial segments of the L_{∞} Walsh arrays. An examination of the tables indicates $E_{n,m}^{i*}$ is maximal for fixed $n + m$ along the line $n = m$ for the first two intervals, and is not far from maximal along this line in the last interval. Tables II, III, and IV reproduce the coefficients for the cases $n = m$, $n = 1, 2, \dots, 7$ for each interval. All coefficients are given to an accuracy slightly greater than that justified by the maximal errors. These errors should not be affected by reasonable rounding.

Although a detailed study of the numerical stability of the approximations has not been made, Horner's method for evaluating the polynomials involved was reasonably stable in the few cases checked. Conversion to J -fraction form resulted in extreme instability.

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