of order m_i, depends only on λ_i and, by (4.6), its (α, β) element is a function of $\alpha - \beta$. If we define $N_i = (e_2, \cdots, e_{m_i}, 0)$, where $I = (e_1, \cdots, e_{m_i})$, then

$$L^{ii} = \sum_{\nu=0}^{m_i-1} \frac{p_i^{(\nu)}(\lambda_i)}{\nu!\, p_i(\lambda_i)} N_i^{\nu}, \tag{4.8}$$

and is a polynomial in N_i. Since J_i is also a polynomial in N_i it must commute with L^{ii}.

The above results were derived for $H \in$ UHM. However, properties (ii) and (iii) generalize immediately to all Hessenberg matrices by the remarks at the beginning of Section 2.

University of California
Berkeley, California

1. J. G. F. Francis, "The QR transformation. I, II," *Comput. J.*, v. 4, 1961/1962, pp. 265–271, 332–345.
2. V. N. Faddeeva, *Computational Methods of Linear Algebra*, Dover, New York, 1959, p. 20.
3. A. S. Householder, *The Theory of Matrices in Numerical Analysis*, Blaisdell, New York, 1964. MR **30** #5475.
4. M. Marcus & H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, Boston, Mass., 1964. MR **29** #112.
5. J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1965. MR **32** #1894.
6. V. Zeipel, "On Determinates, hvars elementer aro Binomialkoefficenter," *Lunds Univ. Asskr.* II, 1865, pp. 1–68.
7. B. Parlett, "Convergence theory for the QR algorithm on a Hessenberg matrix," *Math. Comp.* (To appear.)

An Elimination Method for Computing the Generalized Inverse*

By Leopold B. Willner

0. Notations. We denote by

A an $m \times n$ complex matrix,
A^* the conjugate transpose of A,
$A_j, j = 1, \cdots, n$ the jth column of A,
A^+ the generalized inverse of A [7],
H the Hermite normal form of A, [6, pp. 34–36],
Q^{-1} the nonsingular matrix satisfying

$$H = Q^{-1}A, \tag{1}$$

$e_i, i = 1, \cdots m$ the ith unit vector $e_i = (\delta_{ij})$,
r the rank of A $(= \operatorname{rank} H)$.

1. Method. The Hermite normal form of A is written as

$$H = \begin{bmatrix} B \\ 0 \end{bmatrix} \qquad \text{where } B \text{ is } r \times n. \tag{2}$$

Received July 13, 1966.
* Research supported by the National Science Foundation Grant GP-5230.

Combining (1) and (2) we have:

$$A = QH = [P, R] \begin{bmatrix} B \\ 0 \end{bmatrix} = PB, \tag{3}$$

where $[P, R]$ is the corresponding partition of Q. Having displayed the $m \times n$ matrix A of rank r as a product of the $m \times r$ matrix P and the $r \times n$ matrix B, which are both of rank r, we have as in [4]

$$A^+ = B^+P^+ = B^*(BB^*)^{-1}(P^*P)^{-1}P^* \tag{4}$$

therefore

$$A^+ = B^*(P^*PBB^*)^{-1}P^* \tag{5}$$

and by (3)

$$A^+ = B^*(P^*AB^*)^{-1}P^*. \tag{6}$$

The method can be summarized as follows:

Step 1. Given A obtain H by Gaussian elimination.

Step 2. From H determine P as follows:

The ith column of P, P_i, $i = 1, \cdots, r$ is

$$P_i = A_j \quad \text{if} \quad H_j = e_i, \qquad j = 1, \cdots, n. \tag{7}$$

Step 3. Calculate P^*AB^*.

Step 4. Invert P^*AB^*.

Step 5. Calculate A^+ using (6).

2. Remarks. (i) From (7) we conclude that in order to obtain P it is unnecessary to keep track of the elementary operations involved in finding H, e.g. [5].

(ii) Representation (4), as a computational method, was suggested by Greville [4], Householder [5] and Frame [2]. The novelty of the present paper lies in equation (6) and Step 2 above.

(iii) Like other elimination methods for computing A^+, e.g. [1], the method proposed here depends critically on the correct determination of rank A, e.g. the discussion in [3].

(iv) The advantage of method (6) over the elimination method of [1] is that here the matrix A^*A (or AA^*) is not computed. However, other matrix multiplications are involved in this method.

3. Example. For

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix}$$

we obtain by Gaussian elimination

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = H$$

since $H_2 = e_1$ we have $P_1 = A_2$, and since $H_3 = e_2$ we have $P_2 = A_3$. Hence for

$A = PB$ we have

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix}$$

and

$$P^*AB^* = \begin{bmatrix} 12 & -3 \\ 5 & 0 \end{bmatrix}$$

from which

$$(P^*AB^*)^{-1} = \frac{1}{15}\begin{bmatrix} 0 & 3 \\ -5 & 12 \end{bmatrix}$$

hence

$$A^+ = B^*(P^*AB^*)^{-1}P^* = \frac{1}{15}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 3 \\ -5 & 7 & 2 \\ 5 & -4 & 1 \\ 5 & -4 & 1 \end{bmatrix}.$$

Northwestern University
Evanston, Illinois 60201

1. A. Ben-Israel & S. Wersan, "An elimination method for computing the generalized inverse of an arbitrary complex matrix," *J. Assoc. Comput. Mach.*, v. 10, 1963, pp. 532–537. MR **31** #5318.

2. J. Frame, "Matrix functions and applications. I: Matrix operations and generalized inverses," *IEEE Spectrum*, v. 1, 1964, no. 3, pp. 209–220. MR **32** #1195.

G. Golub & W. Kahan, "Calculating the singular values and pseudo-inverse of a matrix," *J. SIAM Numer. Anal.*, v. 2, 1965, pp. 205–224.

4. T. N. E. Greville, "Some applications of the pseudo-inverse of a matrix," *SIAM Rev.*, v. 2, 1960, pp. 15–22. MR **22** #1067.

5. A. S. Householder, *The Theory of Matrices in Numerical Analysis*, Blaisdell, New York, 1964. MR **30** #5475.

6. M. Marcus & H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, Boston, Mass., 1964. MR **29** #112.

7. R. Penrose, "A generalized inverse for matrices," *Proc. Cambridge Philos. Soc.*, v. 51, 1955, pp. 406–413. MR **16**, 1082.