An Algorithm For Certain Double Sums of Polynomial Series

By R. Langel

1. Introduction and Summary. Smith [1] has derived an algorithm for summing polynomial series which obey a certain recurrence relation. This algorithm was suggested by Clenshaw's [2], [3] technique for summing a Chebyshev series.

This paper describes an algorithm, similar to Smith's, useful in evaluating certain polynominal series where the polynomials are functions of two indices.

2. The New Algorithm. Certain polynomials (e.g. associated Legendre functions) can be shown to satisfy the following recurrence relations:

$$\begin{aligned}
P^{0,0}(\mu) &= \gamma, \\
P^{r,r}(\mu) &= [\beta(\mu) - h_r]P^{r-1,r-1}(\mu), \qquad r \geqq 1, \\
P^{r,q}(\mu) &= \alpha(\mu)P^{r-1,q}(\mu) - K^{r,q}P^{r-2,q}(\mu), \qquad r \geqq 3; \qquad q < r, \\
&= 0, \qquad q > r, \\
P^{1,0}(\mu) &= \alpha(\mu)P^{0,0}, \\
P^{2,0}(\mu) &= \alpha(\mu)P^{1,0}(\mu) - K^{2,0}P^{0,0}(\mu), \\
P^{2,1}(\mu) &= \alpha(\mu)P^{1,1}(\mu).
\end{aligned} \tag{1}$$

In these equations $P^{r,q}(\mu)$ is a polynomial in μ of degree r; q is defined as the 'order' of the polynomial. $\beta(\mu)$ and $\alpha(\mu)$ are arbitrary functions of μ while h_r and $K^{r,q}$ are constant coefficients, independent of μ.

If we have

$$f(\mu) = \sum_{r=0}^{n} \sum_{q=0}^{r} c_{r,q} P^{r,q}(\mu)$$

for some function $f(\mu)$ then (1) can be used to generate the series by well known techniques.

A faster and more accurate method is to define the parameters

$$\begin{aligned}
B_{r,q} &= 0; \qquad q > r, \\
&= 0; \qquad r > n, \\
&= c_{r,q} + \alpha(\mu)B_{r+1,q} - B_{r+2,q}K^{r+2,q}; \qquad q < r, \\
&= c_{r,q} + (\beta(\mu) - h_{r+1})B_{r+1,r+1} + \alpha(\mu)B_{r+1,r} - K^{r+2,r}B_{r+2,r}; \qquad q = r.
\end{aligned} \tag{3}$$

Substituting (1) into (2) and using (3) one can show that

$$f(\mu) = \gamma B_{0,0}$$

3. Error Analysis. Using Smith's method for examining error accumulation, we

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take

$$E_{r,q} = \text{total error in } B_{r,q}\,,$$

$$\epsilon_{r,q} = \text{error introduced in calculation of } B_{r,q} \text{ from previous } B_{r,q}\text{'s.}$$

It is then easy to show that the $E_{r,q}$ obey the same recurrence relation as the quantities $B_{r,q}$ and hence the total error is approximately given by

$$\gamma_0 E_0 \approx \sum_{r=0}^{n} \sum_{q=0}^{r} \epsilon_{r,q} P^{r,\,q}(\mu). \tag{4}$$

This means that the error, $\epsilon_{r,q}$, introduced at the r, q, step contributes $\epsilon_{r,q}P^{r,\,q}$ to the final answer, i.e., that errors do not build up disastrously.

Using techniques similar to those developed by Wilkinson [4], we can show that $\epsilon_{r,q} \approx 2^{-t+1}B_{r,q}$, where t is the number of (binary) bits available on the computer (fraction only for floating point). For floating point operation on an IBM 7094, $t = 27$.

If the summation is carried out in the conventional manner, and if we take

$$E'_{r,q} = \text{total error in } P^{r,\,q}(\mu),$$

$$\epsilon'_{r,q} = \text{error in computing } P^{r,\,q} \text{ from previous } P^{r,\,q}\text{'s.}$$

We obtain

$$\begin{aligned} E'_{r,r} &\approx [\beta(\mu) - h_r]E'_{r-1,r-1} + \epsilon'_{r,r}\,, \\ E'_{r,q} &\approx \alpha(\mu)E'^{\,r-1q} - K^{r,\,q}E'^{\,r-2,\,q} + \epsilon'_{r,q}\,, \end{aligned} \tag{5}$$

where

$$\epsilon'^{\,r,\,q} = 2^{-t+1}P^{r,\,q}.$$

This gives a total error of

$$\sum_{r=0}^{n} \sum_{q=0}^{r} c_{r,q} \cdot E'_{r,q} \tag{6}$$

plus the error in multiplying $c_{r,q}$ by $P^{r,\,q}$. From the form of $E'_{r,q}$ it is apparent that this error accumulates faster than that in our algorithm.

4. Summing the Derivative of the Series. A similar technique can be used to evaluate

$$f'(\mu) = \sum_{r=1}^{n} \sum_{q=0}^{r} c_{r,q} Q^{r,\,q}(\mu), \tag{7}$$

where

$$Q^{r,\,q} = (d/d\,\mu)P^{r,\,q}(\mu).$$

Taking the derivative of (1) gives a recurrence relation for the $Q^{r,\,q}$ in terms of other $Q^{r,\,q}$ and in terms of the $P^{r,\,q}$. Substituting in (7) and using (3) gives

$$f'(\mu) = \alpha'(\mu) \sum_{r=1}^{n} \sum_{q=0}^{r-1} B_{r,q} P^{r-1,\,q}(\mu) + \beta'(\mu) \sum_{r=1}^{n} B_{r,r} P^{r-1,\,r-1}(\mu). \tag{8}$$

The first sum in the derivative can be evaluated by the same process used to find $f(\mu)$ where the $B_{r,q}$ now replaces the $c_{r,q}$. The second sum may be evaluated by an obvious application of the algorithm given by Smith.

5. Advantages and Test Case. A simple computation shows that for $n > 3$ the new algorithm saves about $3n$ multiplications and additions when computing $f(\mu)$ and about $n + \sum_{k=3}^{n} k$ multiplications and $\sum_{k=1}^{n} k$ additions when computing $f'(\mu)$.

This algorithm has been used successfully in a Fortran application on an IBM 7094 using Legendre Polynomials. We found computation time to be reduced by approximately 25 % and accuracy increased by .5 to one order of magnitude.

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A Note on the Summation of the Generalised Hypergeometric Functions

By Arun Verma

1. Introduction. In 1962 MacRobert and Ragab [2], obtained the sum of an infinite series of E-functions as a product of gamma functions. Recently, the author [4] has extended the result to obtain the sum of a series of G-functions as a product of gamma functions. Carlitz [1], in a recent paper obtained some results of a different nature. He obtained the sum of a finite number of terms of hypergeometric series as a product of gamma functions. Slater [3] has also given the sum of a generalised hypergeometric function in terms of elementary functions under a set of conditions.

In this note the sum of certain generalised hypergeometric functions, bilateral hypergeometric functions, generalised basic hypergeometric functions and the generalised basic bilateral hypergeometric functions are deduced.

2. Notation. Let

$$[a]_n = a[a+1][a+2] \cdots [a+n-1]; \qquad [a]_0 = 1,$$

then

$${}_rF_s\begin{bmatrix}(a_r); z\\ (b_s)\end{bmatrix}_N = \sum_{n=0}^{N} \frac{[(a_r)]_n}{[1]_n[(b_s)]_n} z^n.$$

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