A Simple "Filon-Trapezoidal" Rule

By E. O. Tuck

Filon's quadrature is a formula for the approximate evaluation of Fourier integrals such as

$$F(\omega) = \int_{-T}^{T} dt\, e^{i\omega t} f(t), \tag{1}$$

which retains uniform accuracy even when ω is so large that many oscillations of the integrand occur within a given element δt of the range of integration. The original Filon formula [1] was derived on the assumption that $f(t)$, rather than the complete integrand, may be approximated stepwise by parabolas, so that it may be called a 'Filon-Simpson' rule. More sophisticated 'Filon' rules have appeared (e.g. [2], and the references quoted in [2]), but in fact with fast computers it is more useful to go in the other direction, towards the least sophisticated integration formula of all.

The ordinary trapezoidal rule gives as an approximation

$$F(\omega) = \delta t \sum_{n=-N}^{N} w_n e^{i\omega n\delta t} f(n\delta t) \tag{2}$$

$(N\delta t = T)$, with weights

$$\begin{aligned} w_n &= 1, \qquad n \neq \pm N, \\ w_{\pm N} &= \tfrac{1}{2}. \end{aligned} \tag{3}$$

The simplicity of the weights makes this the most desirable formula to use when the number $2N + 1$ of given data values $f(n\delta t)$ is large; for instance the trapezoidal rule is invariably used in power spectral analyses [3]. However, formula (2) cannot be used unless $\omega\delta t \ll 1$, since the whole integrand is supposed to vary linearly over an element δt. But it is an exceedingly simple task to derive a modification to (2) by assuming only that $f(t)$ itself varies linearly over the element δt. The analysis is similar to that used to derive the Filon-Simpson rule, and will not be given here. The resulting integration formula is of the same form as (2), but the weights are now functions of $\omega\delta t$, namely

$$\begin{aligned} w_{-N} &= (1 + i\omega\delta t - e^{i\omega\delta t})/\omega^2\delta t^2, \\ w_n &= ((\sin \tfrac{1}{2}\omega\delta t)/\tfrac{1}{2}\omega\delta t)^2, \qquad n \neq \pm N, \\ w_N &= (1 - i\omega\delta t - e^{-i\omega\delta t})/\omega^2\delta t^2. \end{aligned} \tag{4}$$

Note that the new weights tend to the trapezoidal-rule values (3) as $\omega\delta t \to 0$.

A particular case of interest is when the range of integration $2T$ is infinite. Suppose we define

$$F_{\text{TRAP}} = \delta t \sum_{n=-\infty}^{\infty} e^{i\omega n\delta t} f(n\delta t) \tag{5}$$

as the ordinary trapezoidal-rule approximation for this case. Then the Filon-

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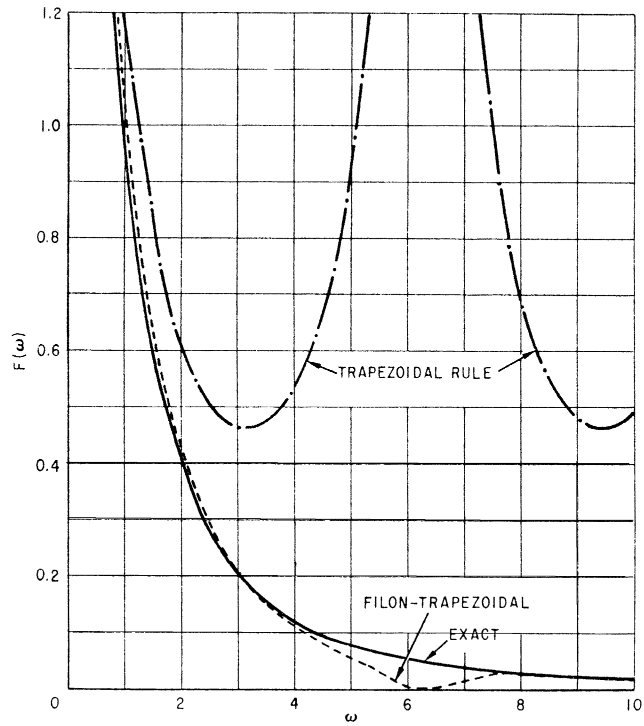

FIG. 1. *Exact and approximate values for the Fourier integral* $F(\omega) = \int_{-\infty}^{\infty} dt \exp(-|t| + i\omega t)$

trapezoidal rule states that an approximation with uniform validity with respect to ω is

$$(6) \qquad F_{\text{FILON}} = ((\sin \tfrac{1}{2}\omega\delta t)/\tfrac{1}{2}\omega\delta t)^2 F_{\text{TRAP}} .$$

That is, the Filon modification is nothing more than a *simple multiplicative factor* applied to the results of the crude trapezoidal rule.

For example, suppose $f(t) = e^{-|t|}$ and $T = \infty$. Then the exact Fourier transform is

$$(7) \qquad F(\omega) = 2/(1 + \omega^2).$$

The series (5) resulting from the application of the ordinary trapezoidal rule maybe summed to give

$$(8) \qquad F_{\text{TRAP}} = \delta t(1 - e^{-2\delta t})/(1 - 2e^{-\delta t} \cos \omega\delta t + e^{-2\delta t}),$$

whence the Filon-modified result follows immediately from (6). Fig. 1 shows the exact, trapezoidal, and Filon-trapezoidal results. In order to exaggerate the differences between the three curves, an absurdly large value $\delta t = 1.0$ has been used for the interval of integration, but even with this coarse subdivision the low-frequency ($\omega < 1$ say) error in the two trapezoidal-rule approximations is less than 8%. Above $\omega = 1$ the trapezoidal rule begins to give ever-decreasing accuracy. Indeed, it is clear from Eq. (5) that the trapezoidal approximation to any Fourier integral is periodic in ω, with period $2\pi/\delta t$ (twice the "Nyquist frequency" [3]) so that the monotone decreasing nature of the true integral (7) cannot be achieved. On the other hand, except for a relatively narrow "dropout" region* near $\omega = 2\pi/\delta t$, the Filon-modified result retains approximately the same 8% accuracy over the whole frequency range shown.

It is clear that the use of the simple factor (6) results in a profound improvement in the approximation for relatively high frequencies. The modification can hardly do harm, and there seems no reason why it could not be employed every time a Fourier integral is to be computed by the trapezoidal rule. If, as in the example given, the function $f(t)$ is mathematically defined, the benefits are immediately obvious, and are available no matter how high the frequency. On the other hand, when $f(t)$ is (say) the autocorrelation function of an experimental record, there are other factors [3] which limit consideration to relatively low frequencies; nevertheless it may well be that if Eq. (6) were used, one could approach a little closer to the Nyquist frequency than has been customary.

Hydromechanics Laboratory
David Taylor Model Basin
Washington, D. C.

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* The width of this region decreases markedly if more realistic small values of δt are used.