1. E. B. Escott, "Amicable numbers," *Scripta Math.*, v. 12, 1946, pp. 61–72. MR **8**, 135.
2. P. Poulet, "Forty-three new couples of amicable numbers," *Scripta Math.*, v. 14, 1948, p. 77.
3. M. Garcia, "New amicable pairs," *Scripta Math.*, v. 23, 1957, pp. 167–171. MR **20** #5158.
4. L. E. Dickson, "Theorems and tables on the sum of the divisors of a number," *Quart. J. Pure Appl. Math.*, v. 44, 1913, pp. 264–296.
5. O. Ore, *Number Theory and Its History*, McGraw-Hill, New York, 1948. MR **10**, 100.
6. L. E. Dickson, *History of the Theory of Numbers*, Carnegie Institution Publ. no. 256, 1919; reprint, G. E. Stechert, 1934.
7. D. H. Lehmer, *Guide to Tables in the Theory of Numbers*, Bull. Nat. Research Council, no. 105, National Research Council, Washington, D. C., 1941. MR **2**, 247.
8. P. Poulet, *La Chasse aux Nombres*, Vol. 1, Stevens, Brussels, 1929.

Primes of the Form $n^4 + 1$

By M. Lal

In this note we report 172 new primes of the form $n^4 + 1$ and tabulate all such primes for $1 \leqq n \leqq 4004$.

Factorization of the numbers of the form $n^4 + 1$ has been extensively studied by Cunningham [1] and Gloden [2], [3]. They used a sieve method based on the four solutions of the congruence equation

$$(1) \qquad x^4 + 1 \equiv 0 \pmod p$$

for all primes of the form $8k + 1$. With primes less than 4×10^6, numbers $n^4 + 1$ for $n \leqq 2000$ have been completely factorized.

For $p > 4 \times 10^6$, it becomes rather difficult and time consuming to solve (1). Consequently, it renders such a sieve less practical. An analysis of the growing inefficiency of such a sieve with increasing p is given in [5, p. 188]. However, the range for n can be extended by using Alway's method [4] of factorization modified for odd divisors of the form $8k + 1$ and testing each number $n^4 + 1$ individually.

With the modified Alway's method, we found all primes for $1 \leqq n \leqq 1000$. This was done to check the program and to provide independent data which is not readily accessible for this interval. The search was then extended to $2000 \leqq n \leqq 4004$; 172 primes and one prime factor for other composite numbers were indentified. The time required to establish the primality of $n = 4002$ is 2.0 hours on the IBM 1620 computer Model II. As the project required several hundred hours of machine time, the search was made on three IBM 1620 computers—one at Memorial and two at Kingston. All primes of the form $n^4 + 1$, complemented by those given in [3] for $1000 < n < 2040$, are presented in Table 1.

Discussion of Results. Shanks [5] has made a conjecture regarding the number of primes $Q(N)$ of the form $n^4 + 1$ for $1 \leqq n \leqq N$ and has given the following expression:

$$(2) \qquad Q(N) \sim .66974 \int_2^N \frac{dn}{\log n}.$$

The observed count and those computed by using (2), rounded to the first decimal, are given below in Table 2.

Received July 15, 1966.

TABLE 1*

Primes of the form $n^4 + 1$

1	48	132	204	276	364	492	566	702	772
2	54	140	210	278	374	494	568	710	778
4	56	142	220	288	414	498	582	730	786
6	74	154	228	296	430	504	584	732	788
16	80	160	238	312	436	516	600	738	798
20	82	164	242	320	442	526	616	742	800
24	88	174	248	328	466	540	624	748	810
28	90	180	254	334	472	550	628	758	856
34	106	194	266	340	476	554	656	760	874
46	118	198	272	352	488	556	690	768	894
912	996	1144	1246	1404	1536	1610	1680	1788	1910
914	1038	1150	1252	1406	1540	1612	1688	1806	1916
928	1042	1152	1270	1428	1542	1618	1700	1820	1926
930	1072	1170	1280	1434	1552	1622	1706	1824	1932
936	1076	1180	1302	1442	1554	1638	1710	1836	1934
952	1088	1200	1322	1446	1558	1644	1718	1850	1942
962	1126	1202	1330	1458	1568	1646	1722	1854	1944
966	1132	1218	1344	1472	1586	1650	1738	1864	1948
986	1136	1236	1382	1486	1594	1652	1754	1870	1952
992	1142	1238	1388	1496	1598	1666	1772	1892	1956
1962	2074	2204	2260	2374	2478	2560	2674	2770	2964
1972	2102	2206	2266	2376	2482	2578	2676	2798	2998
1978	2104	2222	2292	2384	2486	2590	2690	2804	3006
1986	2108	2224	2296	2400	2488	2598	2698	2834	3012
1994	2126	2226	2312	2408	2510	2604	2700	2866	3022
2040	2142	2238	2322	2414	2512	2612	2724	2872	3030
2044	2152	2240	2336	2428	2522	2622	2732	2876	3046
2046	2158	2250	2350	2432	2536	2640	2734	2902	3070
2058	2162	2254	2360	2436	2546	2642	2736	2936	3084
2068	2192	2256	2368	2438	2554	2646	2740	2958	3090
3094	3246	3410	3502	3626	3720	3862	3972		
3100	3254	3416	3516	3632	3752	3870	3982		
3104	3268	3422	3522	3642	3756	3872	3988		
3108	3286	3450	3530	3644	3764	3882	3992		
3124	3288	3456	3550	3666	3780	3896	3998		
3128	3322	3464	3574	3688	3796	3910	4000		
3132	3326	3468	3576	3692	3802	3926	4002		
3162	3378	3472	3586	3700	3842	3954			
3200	3386	3480	3618	3702	3854	3958			
3244	3390	3492	3620	3718	3856	3960			

* I am indebted to the late Professor Albert Gloden for his kind permission to include his results for $1000 < n < 2040$.

The agreement between the actual and computed counts is remarkably good. From Table 1, one observes that "twin" primes (those where $n^4 + 1$ and $(n + 2)^4 + 1$ are both primes) occur quite frequently. The number of such twins $P(N, N + 2)$ for $1 \leqq n \leqq N$ is given in Table 3.

On the basis of heuristic arguments [6], one would expect the number

TABLE 2

N	Counts	
	Observed	Computed
500	63	67.5
1000	111	118.3
1500	150	165.3
2000	205	210.1
2500	254	253.5
3000	292	295.8
3500	330	337.3
4000	376	377.9

TABLE 3
Twin primes

N	$P(N, N + 2)$	$R = N/(\log N)^2$	$P(N, N + 2)/R$
1000	18	20.96	.86
2000	30	34.62	.87
3000	46	46.80	.98
4000	58	58.15	1.00

$P(N, N + 2)$ to be proportional to $N/(\log N)^2$ and it is gratifying to note that this is indicated in column 4 above. These twins are analogous to the Gaussian twin primes examined in [7].

Acknowledgments. I wish to thank the Directors of the Computing Centre of Queen's University and the Royal Military College, Kingston, for the free use of their computers. Thanks are also due to the Canadian Mathematical Congress for the award of a Summer Research Fellowship.

Department of Mathematics
Memorial University of Newfoundland
St. John's, Newfoundland, Canada

1. A. J. C. CUNNINGHAM, *Binomial Factorizations*, Vols. I, IV, Francis Hodgson, London, 1923.
2. A. GLODEN, "Additions to Cunningham's factor table of $n^4 + 1$," *Math. Comp.*, v. 16, 1962, pp. 239–241. MR **26** #2384.
3. A. GLODEN, "New primes of the form $n^4 + 1$," *Math. Comp.*, v. 19, 1965, pp. 144–145. MR **30** #1105.
4. G. G. ALWAY, "A method of factorization using a high-speed computer," *MTAC*, v. 6, 1952, pp. 59–60.
5. D. SHANKS, "On numbers of the form $n^4 + 1$," *Math. Comp.*, v. 15, 1961, pp. 186–189; Corrigendum, *ibid.*, v. 16, 1962, p. 513. MR **22** #10941.
6. P. T. BATEMAN & R. A. HORN, "A heuristic asymptotic formula concerning the distribution of prime numbers," *Math. Comp.*, v. 16, 1962, pp. 363–367. MR **26** #6139.
7. D. SHANKS, "A note on Gaussian twin primes," *Math. Comp.*, v. 14, 1960, pp. 201–203. MR **22** #2586.