Vol. II, 4 + 348 unnumbered pp., 28 cm. Government agencies or their contractors may obtain copies from Defense Documentation Center (DDC), Cameron Station, Alexandria, Va. All others should apply to Clearinghouse for Federal Scientific and Technical Information (CFSTI), Sills Building, 5285 Port Royal Road, Springfield, Va.

Spheroidal wave functions result when the scalar Helmholtz equation is separated in spheroidal coordinates, either prolate or oblate. The angular prolate spheroidal wave functions, for example, satisfy a differential equation of the form

$$\frac{d}{dz}\left[(1 - z^2)\frac{du}{dz}\right] + \left(\lambda_{mn} - c^2z^2 - \frac{m^2}{1 - z^2}\right)u = 0.$$

The solutions of this equation are much more complicated than either Bessel or Legendre functions, in which, in fact, series solutions of the spheroidal functions are most often expanded. The complexity arises from the fact that the spheroidal differential equation has an irregular singular point at ∞ and two regular ones at $z = \pm 1$, in contrast to the three regular ones of the Legendre equation and to the one regular and one irregular singularity of the Bessel equation.

The construction of tables of spheroidal wave functions involves the calculation of the eigenvalues λ_{mn} of the differential equation, that is, those values of λ for which there are solutions that are finite at $z = \pm 1$, and the calculation of the coefficients in expansions in terms of either Legendre or spherical Bessel functions. In the past, such calculations have been, for the most part, sporadic and in many cases not very accurate.

In Volume I the eigenvalues $\lambda_{on}(c)$ and normalization constants $N_{on}(c)$ are tabulated for $c = 0.1(0.1)10.0$ and n ranging from 0 up to a maximum of 20. The $m = 0$ radial functions of the first and second kinds, and their first derivatives, are given for the same c and n for $\xi = 1{\cdot}0000500$, $1{\cdot}0050378$, $1{\cdot}0206207$ and $1{\cdot}1547005$, corresponding to prolate spheroids of length-to-width ratios 100:1, 10:1, 5:1 and 2:1 respectively. Values of the $m = 0$ angular functions and their first derivatives are presented in Volume II for $c = 0.1(0.1)10.0$ with n ranging from 0 up to a maximum of 20, and $\eta = 0(0.05)1.0$.

All computations were carried out on The University of Michigan IBM 7090 computer. The program is described in some detail.

These tables should be very useful for the calculation of various acoustical problems that involve prolate spheroids. In electromagnetic theory, however, only rather simple problems can be treated by means of the functions for $m = 0$. In most electromagnetic problems, the functions with $m = 1$ and often higher values of m are required.

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38[L, X].—L. N. NOSOVA & S. A. TUMARKIN, *Tables of Generalized Airy Functions for the Asymptotic Solution of the Differential Equation* $\epsilon(py')' + (q + \epsilon r)y = f$, translated by D. E. Brown, Pergamon Press, Inc., Long Island, New York, 1965, xxxiv + 89 pp., 25 cm. Price $12.00.

This is a translation from the Russian originally published in 1961. The original version does not seem to have been previously reviewed in these annals although some of the tables have appeared in another work of the authors [1]. Tumarkin [2] studied asymptotic expressions for the solution of the inhomogeneous differential equation

$$\epsilon \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \epsilon r(x)]y = f(x) \tag{1}$$

for small ϵ where $q(x)$ has a simple zero at the origin. It is also assumed that $p(x)$ is positive. By a change of variables, namely

$$y = (p(du/dx))^{1/2}\eta, \tag{2}$$

(1) may be put in the form

$$\epsilon(d^2\eta/du^2) + u\eta = g, \tag{3}$$

where u is expressed by the asymptotic series

$$u \sim \sum_{k=0}^{\infty} \epsilon^k u_k(x). \tag{4}$$

To generate the desired solution of (1) it is shown in [2] and in the present volume that it is sufficient to have tables for the solution of

$$y'' + ty = 1. \tag{5}$$

To describe the functions tabulated, we use the standard notation of [3] and [4]. Solutions of (5) may be taken in the form

$$e_0(t) = (2\pi/3)\, B_i(-t) + \pi\, T_i(-t) = \pi\, H_i(-t), \tag{6}$$

$$\tilde{e}_0(t) = (-\pi/3)\, B_i(-t) + \pi\, T_i(-t) = -\pi\, G_i(-t) \tag{7}$$

so that the Airy integral

$$B_i(-t) = e_0(t) - \tilde{e}_0(t). \tag{8}$$

In the above

$$\pi\, T_i(-t) = \tfrac{1}{2}t^2\, {}_1F_2(1; 4/3, 5/3; -\xi^2/4) = \tfrac{2}{3}t^{1/2}s_{0,1/3}(\xi), \qquad \xi = \tfrac{2}{3}t^{3/2},$$

where $s_{0,1/3}(\xi)$ is a Lommel function, see [4], and the notation $H_i(t)$ and $G_i(t)$ is that introduced by Scorer [5] and used further by Rothman [6, 7]. We also have need for the notation

$$h_n(t) = \xi^{1/3}H_{1/3}^{(n)}(\xi), \qquad n = 1, 2, \tag{9}$$

where $H_{1/3}^{(n)}(z)$ is the usual notation for the Hankel functions. Also

$$e_1(t) = 1 - te_0(t), \qquad e_2(t) = -te_1(t)/2. \tag{10}$$

It is useful to record the integral representations

$$e_0(t) = \int_0^{\infty} \exp(-tx - x^3/3)\, dx, \qquad e_1(t) = \int_0^{\infty} x^2 \exp(-tx - x^3/3)\, dx. \tag{11}$$

Finally, following [4], we can show that

$$(12) \qquad \int_0^t B_i(-x)\,dx = e_0(t)B_i'(-t) - e_0'(t)B_i(-t),$$

$$(13) \qquad \int_0^t A_i(-x)\,dx = \frac{2}{3} + e_0(t)A_i'(-t) - e_0'(t)A_i(-t).$$

A description of the Tables follows.

Table 1.

$$t = is, \qquad e_n(is) = R(e_n) + iI(e_n)$$

$$e_n' = de_n/ds = R(e_n') + iI(e_n')$$

$n = 0$: $0 \leqq s \leqq 8$, 7d; $8 \leqq s \leqq 9$, 9D.

$n = 1$: $0 \leqq s \leqq 6$, 7d; $6 \leqq s \leqq 9$, 7D.

$n = 2$: $0 \leqq s \leqq 3$, 7d; $3 \leqq s \leqq 7$, 6D, $7 \leqq s \leqq 9$, 5D.

Note the convention $e_n' = de_n/ds$ so that when $t = is$, $de_n/dt = -i\, de_n/ds$. This practice is followed throughout.

Table 2.

$$e_n(t),\ e_n'(t)$$

$n = 0, 1, 2$, $0 \leqq t \leqq 10$, $-1 \leqq t \leqq 0$, 7D.

Table 3.

$$\tilde{e}_n(t),\ \tilde{e}_n'(t)$$

$n = 0$: $-10 \leqq t \leqq 0$, $0 \leqq t \leqq 1$, 7D.

$n = 1$: $-6 \leqq t \leqq 0$, $0 \leqq t \leqq 1$, 7D; $-10 \leqq t \leqq -6$, 6D.

$n = 2$: $-3 \leqq t \leqq 0$, $0 \leqq t \leqq 1$, 7D;

$-8 \leqq t \leqq -3$, 6D; $-10 \leqq t \leqq -8$, 5D.

Note on p. 63 for $e_n(t)$ read $\tilde{e}_n(t)$.

Table 4.

$$t = is, \qquad h_n = R(h_n) + iI(h_n),$$

$$h_n' = dh_n/ds = R(h_n') + iI(h_n'),$$

$$n = 1, 2, \qquad s = 0(0.1)6,\ 6\text{D}.$$

These are rounded from the 8D Harvard tables [8]. In the latter $h_n' = dh_n/dt$.

Comparison of the present tables with those in [5]–[8] shows that essentially only Table 1 is new. See [9] for tables of (12–13).

The introduction describes the method of calculation, checks used, and ranges for which the tables may be linearly interpolated.

Y. L. L.

1. L. N. Osipova & S. A. Tumarkin, *Tables for the Calculation of Toroidal Shells*, Akad. Nauk SSSR, Moscow, 1963. (See *Math. Comp.*, v. 18, 1964, pp. 677–678.)
2. S. A. Tumarkin, "Asymptotic solution of a linear nonhomogeneous second order differential equation with a transition point and its application to the computations of toroidal shells and propeller blades," *Prikl. Mat. Meh.*, v. 23, 1959, pp. 1083–1094; English transl., *J. Appl. Math. Mech.*, v. 23, 1959, pp. 1549–1565.
3. Milton Abramowitz & Irene A. Stegun, Editors, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Applied Mathematics Series No. 55, U. S. Government Printing Office, Washington, D. C., 1964. (See *Math. Comp.*, v. 19, 1965, pp. 147–149.)
4. Y. L. Luke, *Integrals of Bessel Functions*, McGraw-Hill, New York, 1962. (See *Math. Comp.*, v. 17, 1963, p. 318–320.)
5. R. S. Scorer, "Numerical evaluation of integrals in the form $I = \int_{x_1}^{x_2} f(x)e^{i\varphi(x)}dx$ and the tabulation of the function $G_i(z) = 1/\pi \int_0^\infty \sin(uz + u^3/3)\, du$", *Quart. J. Mech. Appl. Math.*, v. 3, 1950, pp. 107–112. (See *MTAC*, v. 4, 1950, p. 215.)
6. M. Rothman, "The problem of an infinite plate under an inclined loading with tables of the integrals of $A_i(\pm x)$ and $B_i(\pm x)$," *Quart. J. Mech. Appl. Math.*, v. 7, 1954, pp. 1–7. (See *MTAC*, v. 8, 1954, p. 162.)
7. M. Rothman, "Tables of the integrals and differential coefficients of $G_i(x)$ and $H_i(-x)$," *Quart. J. Mech. Appl. Math.*, v. 7, 1954, p. 379–384. (See *MTAC*, v. 9, 1955, pp. 77–78. On the latter pages are descriptions of further tables related to Airy functions and their integrals. See also [3].)
8. Harvard University Computation Laboratory, Annals, Vol. 2, *Tables of the Modified Hankel Functions of Order One-Third and Their Derivatives*, Harvard Univ. Press, Cambridge, Mass., 1945. (See *MTAC*, v. 2, 1946, pp. 176–177.)
9. K. Singh, J. F. Lumley & R. Betchov, *Modified Hankel Functions and their Integrals to Argument* 10, Engineering Research Bulletin B-87, The Pennsylvania State University, University Park, Penn., 1963. (See *Math. Comp.*, v. 18, 1964, p. 522.)

39[L, X].—F. Tölke, *Praktische Funktionenlehre*, Band II: *Theta-Funktionen und spezielle Weierstrasssche Funktionen*, Springer-Verlag, Berlin, 1966, vii + 248 pp., 28 cm. Price DM 84.

This is the first part of a monumental work on theta functions and elliptic functions. It contains an incredible wealth of formulas and theorems involving the elliptic theta functions and those of Weierstrass' elliptic functions which have periods 1 and $i\kappa$ or 1 and $\frac{1}{2} + i\kappa/2$. Four future volumes will treat the Jacobi elliptic functions, special Weierstrass Zeta and Sigma functions, elliptic integrals and Jacobi elliptic functions in the complex domain, general Weierstrass elliptic functions and derivatives with respect to the parameter, integrals of Theta functions and bilinear expansions. The final volume will contain numerical tables.

The present volume consists of four chapters and altogether 107 sections. Everything that can be expected to be helpful to the applied mathematician is derived briefly and stated in full detail, including approximation formulas for the parameter functions which are correct up to the fifth decimal. Partial differential equations, derivatives, values for specialized arguments and addition theorems for the Theta functions are given in great detail. Many expansions are given with a large number of numerical coefficients. The Weierstrass elliptic integrals in normal form (of the 2nd and 3rd kinds) are expressed in terms of the logarithms of the Theta functions. There are 765 numbered formulas, but many of them are groups of formulas. To the best of the knowledge of the reviewer, nothing comparable in completeness and abundance of details exists in the literature.

There is no index, but the table of contents provides a very good orientation. The list of references consists of 147 entries; however, the *Handbook of Elliptic Integrals for Engineers and Physicists* by P. F. Byrd and M. D. Friedman, Springer, Berlin, 1954, is missing, probably because it has little in common with the present work.

This is certainly no textbook, but it is a very valuable source of information for