Rational Approximations to the Incomplete Elliptic Integrals of the First and Second Kinds*

By Wyman G. Fair and Yudell L. Luke

In this note we derive rational approximations (in Eqs. (20) and (21) below) to the integrals

$$F(\varphi, k) = \int_0^\varphi (1 - k^2 \sin^2 t)^{-1/2} dt, \tag{1}$$

and

$$E(\varphi, k) = \int_0^\varphi (1 - k^2 \sin^2 t)^{1/2} dt, \tag{2}$$

where k^2 is real and $0 < \varphi < \pi/2$, by obtaining the main diagonal Padé approximations to closely related functions. It is sufficient to consider the case $0 < k^2 < 1$, for if $k^2 > 1$,

$$\begin{gathered} F(\varphi, k) = k_1 F(\beta_1, k_1) \quad \text{and} \quad E(\varphi, k) = k_1[E(\beta_1, k_1) + (1 - k^2)^2 F(\beta_1, k_1)], \\ k_1 = 1/k \quad \text{and} \quad \beta_1 = \arcsin (k \sin \varphi), \end{gathered} \tag{3}$$

while if $k^2 < 0$,

$$\begin{gathered} F(\varphi, k) = (1 - k_2^2)^{1/2} F(\beta_2, k_2) \quad \text{and} \\ E(\varphi, k) = (1 - k_2^2)^{-1/2}\left[E(\beta_2, k_2) - \frac{k_2^2 \sin \beta_2 \cos \beta_2}{(1 - k_2^2 \sin^2 \beta_2)^{1/2}}\right], \\ k_2 = |k|(1 - k^2)^{-1/2} \quad \text{and} \quad \beta_2 = \arcsin\left[\left(\frac{1 - k^2}{1 - k^2 \sin^2 \varphi}\right)^{1/2} \sin \varphi\right]. \end{gathered} \tag{4}$$

Define $m = k^2$ and

$$\begin{gathered} a = \left[\frac{(2 - m)^2}{1 + m}\right]^{1/3} > 0, \qquad b = \left[\frac{(1 - 2m)^3}{(m - 2)(m + 1)}\right]^{1/3}, \\ c = \left[\frac{(1 + m)^2}{m - 2}\right]^{1/3} < 0, \qquad x = c + \frac{a - c}{\sin^2 \varphi}, \\ h = a\left[c + \frac{b(2m - 1)}{m - 2}\right] < 0, \quad g = 2m - 1, \qquad s = 2\left[\frac{2 - m}{3a}\right]^{1/2}, \\ r(x) = x^3 + hx + g, \qquad v(x) = \frac{(x - c)^3 (x - a)}{x - b}, \\ I_1(x) = \int_x^\infty [r(t)]^{-1/2} dt \quad \text{and} \quad I_2(x) = \int_x^\infty [v(t)]^{-1/2} dt. \end{gathered} \tag{5}$$

Received February 3, 1967.

* This work was supported by the National Aeronautics and Space Administration under Contract No. NASA Hq. 80X0108(64).

Then $a > b > c$ are the real roots of $r(z) = 0$ and it follows from [1] that

(6) $$F(\varphi, k) = s^{-1}I_1(x) \quad \text{and} \quad E(\varphi, k) = s^{-1}I_2(x) .$$

Set

(7) $$G_1(x) = [r(x)]^{1/2}I_1(x) , \qquad G_2(x) = \frac{[v(x)]^{1/2}}{2x} I_2(x) .$$

Then $G_l(x)$ $(l = 1, 2)$ satisfies the differential equation

(8) $$r(x)\gamma_l(x)G_l'(x) - \delta_l(x)G_l(x) + r(x) = 0 ,$$

where

$$\gamma_1(x) = 1 , \qquad \gamma_2(x) = 2x , \qquad \delta_1(x) = \tfrac{1}{2}(3x^2 + h) ,$$
$$\delta_2(x) = x^3 - 2(a + 2b)x^2 + (ab - bc - 3ac)x + 2abc .$$

For convenience, we make the transformations

(9) $$z = 1/x , \qquad G_1(z) = x^{-1}[2 + x^2H_1(x)] , \qquad G_2(z) = H_2(z) .$$

Then (8) becomes

(10) $$\eta_l(x)H_l'(x) + \rho_l(x)H_l(x) + \xi_l(x) = 0 , \qquad l = 1, 2 ,$$

where

$$\eta_1(x) = x(1 + hx^2 + gx^3) , \qquad \eta_2(x) = 2\eta_1(x) , \qquad \rho_1(x) = \frac{5}{2} + \frac{3h}{2}x^2 + gx^3 ,$$
$$\rho_2(x) = 1 - 2(a + 2b)x + (ab - bc - 3ac)x^2 - 2gx^3 , \qquad \xi_1(x) = -2h - 3gx ,$$
$$\xi_2(x) = -1 - hx^2 - gx^3 , \qquad H_1(0) = \frac{4h}{5} \quad \text{and} \quad H_2(0) = 1 .$$

Main diagonal Padé approximations for the solution to (10) are readily computed by using the results of [2]. For completeness we list the recurrence relations which determine the main diagonal Padé approximations to $H_l(x)$, $l = 1, 2$. In the notation of [2], we have: for $l = 1$;

(11) $$\begin{aligned} y(x) &= H_1(x) , \\ y_0 &= y(0) = 4h/5 , \\ p_0 &= p_2 = 0 , \quad p_1 = 1 , \quad p_3 = h , \quad p_4 = g , \\ q_0 &= 5/2 , \quad q_1 = 0 , \quad q_2 = 3h/2 , \quad q_3 = g , \\ s_0 &= -2h , \quad s_1 = -3g , \quad s_2 = s_3 = 0 , \end{aligned}$$

and for $l = 2$, $y(x) = H_2(x)$,

(12) $$\begin{aligned} y_0 &= y(0) = 1 , \\ p_0 &= p_2 = 0 , \quad p_1 = 2 , \quad p_3 = 2h , \quad p_4 = 2g , \\ q_0 &= 1 , \quad q_1 = -2(a + 2b) , \quad q_2 = ab - bc - 3ac , \quad q_3 = -2g , \\ s_0 &= -1 , \quad s_1 = 0 , \quad s_2 = -h , \quad s_3 = -g . \end{aligned}$$

Let

$$y_n = \frac{A_n}{B_n}, \qquad A_n = \sum_{k=0}^{n} a_{n,k}x^k, \qquad B_n = \sum_{k=0}^{n} b_{n,k}x^k \tag{13}$$

be the nth-order main diagonal Padé approximations to $y(x)$. Then A_n and B_n satisfy

$$A_n = (1 + \beta_n x)A_{n-1} + \alpha_n x^2 A_{n-2}. \tag{14}$$

The equations which determine α_n and β_n are

$$\alpha_n = -\tau_{n-1,1}\left[(-1)^n \alpha_{n-1,1}p_1 + \alpha_{n-1,2}u_1 + 2\sum_{j=3}^{n} \alpha_{n-1,j}\tau_{j-2,1}\right]^{-1}, \tag{15}$$

and

$$\beta_n = -\left[\tau_{n-1,2} + \alpha_{n,2}u_2 + 2\sum_{j=3}^{n} \alpha_{n,j}(\tau_{j-2,2} + \beta_{j-1}\tau_{j-2,1})\right]$$
$$\times \left[2\tau_{n-1,1} + \alpha_{n,2}u_1 + 2\sum_{j=3}^{n} \alpha_{n,j}\tau_{j-2,1}\right]^{-1}, \qquad n = 2, 3, 4 \cdots,$$

where

$$\begin{aligned} \tau_{n,k} = {} & \tau_{n-1,k+2} + 2\beta_n\tau_{n-1,k+1} + \alpha_n^2\tau_{n-2,k} + \beta_n^2\tau_{n-1,k} + (-1)^n\alpha_{n,1}p_{k+2} \\ & + \alpha_{n,2}u_{k+2} + \alpha_{n,2}\beta_n u_{k+1} \\ & + 2\sum_{j=3}^{n} \alpha_{n,j}[\tau_{j-2,k+2} + (\beta_n + \beta_{j-1})\tau_{j-2,k+1} + \beta_n\beta_{j-1}\tau_{j-2,k}], \\ & \qquad n = 2, 3, 4, \cdots, k = 1, 2, 3, \\ u_k = {} & 2y_0q_k + 2s_k + (a_{1,1} + b_{1,1}y_0)q_{k-1} + 2b_{1,1}s_{k-1} \qquad k = 1, 2, 3, 4, \\ \alpha_{k,j} = {} & \alpha_k\alpha_{k-1}\cdots\alpha_j,\ \alpha_{k,k} = \alpha_k,\ \alpha_{k-1,k} = 1 \quad \text{and} \quad \alpha_{k,j} = 0, \qquad k < j - 1. \end{aligned} \tag{16}$$

The starting values for computation are

$$\begin{aligned} \tau_{0,k} &= y_0q_k + s_k, \\ \tau_{1,k} &= -\alpha_1p_{k+2} + y_0q_{k+2} + s_{k+2} + (a_{1,1} + b_{1,1}y_0)q_{k+1} \\ &\quad + 2b_{1,1}s_{k+1} + a_{1,1}b_{1,1}q_k + b_{1,1}^2s_k, \qquad k = 1, 2, 3 \end{aligned}$$

for $l = 1$,

$$\begin{aligned} \alpha_1 &= -6g/7, \qquad \beta_1 = b_{1,1} \\ a_{1,1} &= 6g/7 + \frac{56h^3}{225g}, \qquad b_{1,1} = \frac{14h^2}{45g}; \end{aligned} \tag{17}$$

for $l = 2$,

$$\begin{aligned} \alpha_1 &= -2/3(a + 2b), \qquad \beta_1 = b_{1,1}, \\ a_{1,1} &= \frac{4a^2 + 16b^2 + 25ab - 9bc - 27ac - 9h}{30(a + 2b)}, \\ b_{1,1} &= \frac{-4a^2 - 16b^2 - 19ab + 3bc + 9ac - 3h}{10(a + 2b)}. \end{aligned} \tag{18}$$

In either case, we have

(19) $A_0 = y_0, \quad A_1 = y_0 + a_{1,1}x, \quad B_0 = 1, \quad B_1 = 1 + b_{1,1}x.$

Thus, rational approximations to the incomplete elliptic integrals of the first and second kind respectively are

$$F_n(\varphi, k) = \frac{[r(x)]^{1/2}}{s}\left[2x + \frac{A_n(1/x)}{xB_n(1/x)}\right] \tag{20}$$

and

$$E_n(\varphi, k) = \frac{2x[v(x)]^{-1/2}}{s}\frac{A_n(1/x)}{B_n(1/x)}. \tag{21}$$

In the special case, $k^2 = m = \frac{1}{2}$, the approximation (20) does not apply. However, since $g = 0$ in this case, (20) becomes

(22) $t(1 + ht)H_1'(t) + \frac{1}{4}(5 + 3ht)H_1(t) - h = 0, \quad H_1(0) = 4h/5, \quad t = x^2,$

and $H_1(t) = (4h/5)\ {}_2F_1(1, 3/4; 9/4; -ht)$ is the solution to (22). Padé approximations to this hypergeometric function together with an error analysis are available in [3].

Numerical results indicate rapid convergence of the approximations (20) and (21). These approximations are evidently insensitive to changes in k^2 and are very powerful for $\varphi < \pi/3$. They weaken as φ approaches $\pi/2$; however, the Landen transformations

$$F(\varphi, k) = \frac{2}{1+k}F(\varphi_1, k_1),$$

$$E(\varphi, k) = (1 + k)E(\varphi_1, k_1) + (1 - k)F(\varphi_1, k_1) - k \sin \varphi,$$

where

(23) $k_1 = 2\sqrt{k}/(1 + k)$ and $\varphi_1 = \frac{1}{2}\varphi + \frac{1}{2} \arcsin (k \sin \varphi),$

should reduce φ to the desirable range in all but the extreme cases. For example, if $k = \frac{1}{2}$ and $\varphi = \pi/2$ we have

$$F(\tfrac{1}{2}, \pi/2) = \tfrac{4}{3}F(2\sqrt{2}/3, \pi/3). \tag{24}$$

The approximations $\frac{4}{3}F_n(2\sqrt{2}/3, \pi/3)$ are listed in Table I.

TABLE I

n	$\frac{4}{3}F_n$
4	1.68579 32446
6	1.68575 05579
8	1.68575 03557
10	1.68575 03548
12	1.68575 03548

The true value is 1.68575 03548.

We present in Table II a tabulation of $\epsilon_n = |F(\varphi, k) - F_n(\varphi, k)|$ for a number of values of n, φ and k. The behavior of the error involved in approximating $E(\varphi, k)$

TABLE II

$k^2 = .25$

$\varphi \diagdown n$	4	6	8	10	20
60°	1.89 (−5)*	4.1 (−7)	4.0 (−8)		
80°	7.33 (−2)	2.11 (−2)	7.11 (−3)	3.84 (−3)	8.1 (−7)

by $E_n(\varphi, k)$ is almost identical and so is omitted. In both tables $\epsilon_n < 1.0 \times 10^{-8}$ for $\varphi \leq 30°$ and $n \geq 4$ (k arbitrary) so that these values are not listed. No entry in the table signifies an error less than 1.0×10^{-8}.

$k^2 = .75$

$\varphi \diagdown n$	4	6	8	10	20
60°	1.92 (−3)	4.7 (−7)			
80°	1.51 (−1)	2.55 (−2)	5.44 (−3)	1.13 (−3)	8.0 (−7)

* The number in parentheses indicates the power of ten by which the tabular entry is to be multiplied.

The authors thank John Nelson who wrote the FORTRAN program for the computations. This program is available from the authors on request.

Midwest Research Institute
Kansas City, Missouri 64110

1. P. F. BYRD & M. D. FRIEDMAN, *Handbook of Elliptic Integrals for Engineers and Physicists*, Die Grundlehren der mathematischen Wissenschaften, Band 67, Springer-Verlag, Berlin, 1954. MR **15**, 702.

2. W. FAIR, "Padé approximation to the solution of the Riccati equation," *Math. Comp.*, v. 18, 1964, pp. 627–634. MR **29** #6630.

3. Y. L. LUKE, "The Padé table and the τ-method," *J. Math. Phys.*, v. 37, 1958, pp. 110–127. MR **20** #5558.