Value $\times 10^{34}$	Evaluation of $K_{i50}(20)$ v	B	D
$-.723828$	-1.00	3.65	.0025
.269938	-1.20	4.00	"
.274077	-1.40	4.80	"
.274078	-1.48	5.45	"
"	-1.50	5.60	.005
"	-1.52	6.00	"
"	-1.54	6.50	"

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Mixed Algebraic-Exponential Interpolation Using Finite Differences

By J. W. Layman

The use of finite differences in exponential polynomial interpolation was introduced in [1], where an algorithm was developed which triangularizes the system of equations that determines the coefficients in the interpolating exponential polynomial. In the present note we show that a similar finite-difference algorithm also exists for interpolation by a mixed algebraic-exponential polynomial of the form

$$P(x) = \sum_{n=1}^{N} \sum_{m=0}^{m_n} a_{nm} x^{(m)} n^x \tag{1}$$

for $x = 0, 1, 2, \cdots, \sum_{n=1}^{N} (m_n + 1) - 1$. The symbol $x^{(m)}$ represents the factorial power function $x(x-1)\cdots(x-m+1)$.

We require the basic difference operations E and Δ and, in addition, the diagonal difference S defined by $Sf(x) = \Delta^x f(0)$. The diagonal difference is more precisely defined in [1] and certain difficulties in interpretation are resolved there. These arise when taking higher-order diagonal differences by iteration, $S^n f(x) = SS^{n-1}f(x)$.

The following properties and formulas involving the diagonal-difference opera-

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tion follow more or less directly from the definition. Proofs are left to the reader.

$$\begin{aligned}
S[af(x) + bg(x)] &= aSf(x) + bSg(x)\ , \\
S^m f(x) &= (E - m)^x f(0)\ , \\
S^n[x^{(r)} f(x)] &= x^{(n)} E^{-r} S^n f(k + r)\ , \\
S^m[n^x f(x)] &= n^x S^{m/n} f(x), \text{ if } n \text{ divides } m\ , \\
S[x^{(r)} a^x] &= x^{(r)} a^r (a - 1)^{x-r}\ .
\end{aligned}$$

For consistency we define $x^{(r)}0^{x-r}$ to be zero for $x < r$, $r!$ for $x = r$, zero for $x > r$.

Rather than developing the triangularization procedure for the general algebraic-exponential polynomial in (1), we will restrict ourselves to the special case of $N = 2$ with $m_1 = m_2 = 2$. Then we may write

$$P(x) = a_1 + b_1 x + c_1 x^{(2)} + (a_2 + b_2 x + c_2 x^{(2)})2^x\ . \tag{2}$$

We now apply the operators E and S in the appropriate sequence so that the coefficients are eliminated one-by-one from left to right to obtain the following:

$$\begin{aligned}
f(x) &= a_1 + b_1 x + c_1 x^{(2)} + (a_2 + b_2 x + c_2 x^{(2)})2^x\ , \\
Sf(x) &= a_1 0^x + b_1 x 0^{x-1} + c_1 x^{(2)} 0^{n-2} + a_2 + 2b_2 x + 2^2 c_2 x^{(2)}\ , \\
ESf(x) &= b_1(x + 1)0^x + c_1(x + 1)^{(2)}0^{x-1} + 2b_2(x + 1) + 4c_2(x + 1)^{(2)}\ , \\
E^2 Sf(x) &= c_1(x + 2)^{(2)}0^x + 2b_2(x + 2) + 4c_2(x + 2)^{(2)}\ , \\
E^3 Sf(x) &= a_2 + 2b_2(x + 3) + 4c_2[x^{(2)} + 8x + 12]\ , \\
SE^3 Sf(x) &= a_2 0^x + 2b_2 x 0^{x-1} + 6b_2 0^x + 4c_2[(x + 1)^{(2)} \\
&\qquad + 6x0^{x-1} + 6 \cdot 0^x]\ , \\
ESE^3 Sf(x) &= 2b_2(x + 1)0^x + 4c_2[(x + 1)^{(2)}0^{x-1} \\
&\qquad + 6(x + 1)0^x]\ , \\
E^2 SE^3 Sf(x) &= 4c_2(x + 2)^{(2)}0^x\ .
\end{aligned}$$

This system is triangular for any x; however, the smallest number of data points is required if we take $x = 0$. Several redundant equations can be eliminated to yield the following system:

$$\begin{aligned}
Sf(0) &= a_1 + a_2\ , \\
ESf(0) &= b_1 + a_2 + 2b_2\ , \\
E^2 Sf(0) &= 2c_1 + a_2 + 4b_2 + 8c_2\ , \\
SE^3 Sf(0) &= a_2 + 6b_2 + 24c_2\ , \\
ESE^3 Sf(0) &= 2b_2 + 24c_2\ , \\
E^2 SE^3 Sf(0) &= 8c_2\ .
\end{aligned} \tag{3}$$

For any given instance of the general algebraic-exponential polynomial we proceed in a similar manner. We first apply the operator S, followed by $n + 1$ successive applications of E, where n is the degree of $A_1(x)$, then again apply S, followed by $m + 1$ successive applications of E where m is the degree of $A_2(x)$, $\cdots$ etc. We evaluate these derived polynomials at $x = 0$ and discard redundant equations to obtain the appropriate triangularization scheme.

Consideration of the diagonal-difference operation S shows that $Sf(x)$ is simply the diagonal of so-called leading differences, i.e., the diagonal of differences which passes through $f(0)$. The function $SE^3Sf(x)$, for example, is then obtained by calculating the diagonal of leading differences of $E^3Sf(x)$, that is, the diagonal of differences of $Sf(x)$ through $Sf(3)$. A typical example is shown below.

x	$f(x)$						$Sf(x)$			
0	1						1			$1 = Sf(0)$
1	-1	-2					-2			$-2 = ESf(0)$
2	2	3	5				5			$5 = E^2Sf(0)$
3	1	-1	-4	-9			-9			$-9 = SE^3Sf(0)$
4	-8	-9	-8	-4	5		5	14		$14 = ESE^3Sf(0)$
5	3	11	20	28	32	27	27	22	8	$8 = E^2SE^3Sf(0)$

Substituting the values of $Sf(0), \cdots, E^2SE^3Sf(0)$ from the tabulation into (3) yields a triangular system which is easily solved to obtain:

$$a_1 = 4, \qquad b_1 = 11, \qquad c_1 = 10,$$
$$a_2 = -3, \qquad b_2 = -5, \qquad c_2 = 1.$$

Hence the algebraic-exponential polynomial of the form given in Eq. (2) which fits the given data is

$$P(x) = 4 + 11x + 10x^{(2)} + [-3 - 5x + x^{(2)}]2^x .$$

The principal advantage of the present method is just that of any finite-difference interpolation method, that is, the systematic handling of the given data. For example, Gauss elimination can be used to give, as the fifth equation of (3),

$$2b_2 + 24c_2 = f_4 - 5f_3 + 9f_2 - 7f_1 + 2f_0 ,$$

which the present method gives as $2b_2 + 24c_2 = ESE^3Sf_0$, a result which clearly indicates a systematic difference-table computation procedure. Furthermore, an analysis of the numerical example shows that Gauss elimination requires 15 additions (subtractions) and 12 multiplications involving the given data $f_0, f_1, \cdots, f_5$, to triangularize the coefficient matrix, whereas the present method requires 18 subtractions.

It may be pointed out that the present method leads to a strictly diagonal system when no exponential factors a^x appear, equivalent to the Gregory-Newton Forward Interpolation method. When no algebraic factors appear we obtain the exponential polynomial method of [1].

For a remainder analysis, when approximating functions of known properties, see a paper by Gori [2].

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