Lal's Constant and Generalizations

By Daniel Shanks

1. Introduction. As is known, if $P_1(N)$ is the number of primes of the form $n^2 + 1$ for $1 \leqq n \leqq N$ and, if

$$\mathrm{li}(N) = \int_2^N dn/\log n\,,$$

then *empirically* and *heuristically*, one has, cf. [1]:

$$P_1(N) \sim 0.68641\ \mathrm{li}(N)\,. \tag{1}$$

Thus, while the much weaker proposition

$$P_1(N) \to \infty \tag{2}$$

has still not been proven, the investigation of (1) provides strong evidence that (2) is true.

The case is strengthened when it is realized that even *subsets* of the set of primes $n^2 + 1$ have counts that $\to \infty$ according to well-known laws. For example, if $n = m^2$, we find empirically and heuristically that if $Q_1(N)$ is the number of primes $m^4 + 1$ with $1 \leqq m \leqq N$, then, [2]:

$$Q_1(N) \sim \tfrac{1}{4}\, s_1\, \mathrm{li}(N) = 0.66974\ \mathrm{li}(N)\,. \tag{3}$$

Or, again, if $(m - 1)^2 + 1$ and $(m + 1)^2 + 1$ are both prime, and if $g(N)$ is the number of such pairs for $m + 1 \leqq N$, then, [3]:

$$g(N) \sim 0.48762\ \mathrm{li}_2(N) \tag{4}$$

where

$$\mathrm{li}_2(N) = \int_2^N dn/\log^2 n\,.$$

Recently [4], Lal has counted pairs of primes $(m - 1)^4 + 1$ and $(m + 1)^4 + 1$ —the "conceptual intersection" of both previous subsets—and he found that if the number of such pairs with $m + 1 \leqq N$ is $h(N)$, then $h(N)$ is at least roughly proportional to $\mathrm{li}_2(N)$. (The qualification "at least roughly" has reference to the fact that he went to only $N = 4000$, and $h(4000)$ is only equal to 57.) Lal did not evaluate the constant λ in

$$h(N) \sim \lambda\, \mathrm{li}_2(N)\ ; \tag{5}$$

we do that here.

2. Impossible Congruences. From Bateman's formula [5, Eq. (1)] we find from (3) and (5) that

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$$\lambda = (s_1/4)^2 \cdot 2 \cdot \prod_{p=8k+1} \frac{p(p-\omega(p))}{(p-4)^2} \tag{6}$$

with the product taken over all primes $p = 8k + 1$, and where $\omega(p)$ is the number of residue classes (mod p) that satisfy

$$[(m-1)^4 + 1] \cdot [(m+1)^4 + 1] \equiv 0 \pmod{p} . \tag{7}$$

Now each factor on the left of (7) has four roots, and therefore $\omega(p) = 8$ for each $p = 8k + 1$, provided that

$$A = (m-1)^4 + 1 \equiv 0 \pmod{p}, \qquad B = (m+1)^4 + 1 \equiv 0 \pmod{p} \tag{8}$$

are not *simultaneously* satisfied for any m. This condition is certainly not obvious; it means that no two of the fourth-roots of -1 (mod p) can differ by 2. But if $A \equiv B \equiv 0$ (mod p) then $32m^2(A + B) - 8m(A - B) - (A - B)^2 = 192m^4 \equiv 0$ (mod p). This implies that $m \equiv 0$ (mod p), which contradicts (8). Therefore

$$\lambda = (s_1/4)^2 \cdot 2 \cdot C \tag{9}$$

with

$$C = \prod_{p=8k+1} \frac{p(p-8)}{(p-4)^2} . \tag{10}$$

3. An Infinite Product. To evaluate (10) we want two cases of the following LEMMA. *If*

$$1 - 2kx = \prod_{s=1}^{\infty} \left(\frac{1-x^s}{1+x^s}\right)^{b(k,s)} \tag{11}$$

then

$$b(k, s) = \frac{1}{2s} \sum_{d} \mu(d) (2k)^{s/d} , \tag{12}$$

the sum taken over all odd divisors d of s with $\mu(d)$ the Möbius function. Conversely, the right side of (11) *converges to the left side if* $|2kx| < 1$.

Comments. The lemma for $k = 1$ is proven in [1, p. 322], and the proof of the generalization here is virtually the same. The lemma for $k = 2$ was given without proof in [2, Eq. (5)]. The general lemma is similar to, but not the same as, *Witt's Formula*, cf. [6].

If we now take $x = 1/p$ and $k = 4$ and 2, (10), (11) and (12) give us

$$C = \prod_{p=8k+1} \left(\frac{p^2-1}{p^2+1}\right)^8 \left(\frac{p^3-1}{p^3+1}\right)^{64} \left(\frac{p^4-1}{p^4+1}\right)^{448} \left(\frac{p^5-1}{p^5+1}\right)^{3072} \cdots . \tag{13}$$

4. Lal's Constant and Generalizations. But for $s > 1$ we have [2, Eq. (6)]:

$$\prod_{p=8k+1} \left(\frac{p^s-1}{p^s+1}\right)^2 = \frac{L_{-1}^2(2s)}{L_{-1}(s)L_1(s)L_{-2}(s)L_2(s)} \tag{14}$$

so the factors on the right of (13) are given in terms of the known [7] Dirichlet series $L_a(s)$. As is usual in such calculations we may obtain much faster convergence by computing the first f factors in (10) *directly*—that is, by computing

$$p(p-8)/(p-4)^2 \text{ for } p = 17, 41, 73, 89, \cdots,$$

and then compensating by utilizing instead of (14) the modified

$$(15) \qquad \prod_{p>p_f} \left(\frac{p^s-1}{p^s+1}\right)^2 = \frac{L_{-1}^2(2s)}{L_{-1}(s)L_1(s)L_{-2}(s)L_2(s)} \prod_{p \leqq p_f} \left(\frac{p^s+1}{p^s-1}\right)^2.$$

We therefore obtain

$$(16) \qquad C = 0.88307$$

and, from (3) and (9),

$$(17) \qquad \lambda = 0.79220.^*$$

It is clear that the same techniques may be used in evaluating a large class of constants. Let

$$(18) \qquad c = \prod_{p=Ak+B} \prod_{i=1}^{r} \frac{p-2k_i}{p-2t_i}$$

with $A = 2, 4, 6, 8, 12$, or 24, with B prime to A, and with

$$(19) \qquad \sum_{i=1}^{r} (k_i - t_i) = 0.$$

This generalizes our (10), in which $A = 8$, $B = 1$, $k_1 = 0$, $k_2 = 4$, $t_1 = t_2 = 2$. By use of the Lemma, and products analogous to (14) for other A and B, all such constants (18) are computable without undue difficulty. The needed products

$$\prod_{p=Ak+B} \left(\frac{p^s-1}{p^s+1}\right)$$

can all be evaluated in terms of $L_a(s)$, where the subscripts a are some, or all, of the twelve divisors of 18: ± 1, ± 2, ± 3, ± 6, ± 9, and ± 18. All these $L_a(s)$ are known [7]. If one or more primes $p = Ak + B$ in (18) are less than one or more $2k_i$ or $2t_i$, then clearly one must choose the p_f above to be greater than all such p, otherwise the sequence analogous to (13) will diverge. By increasing p_f, one first eliminates any divergence, and then obtains more rapid convergence.

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* One has $\lambda \operatorname{li}_2(4000) = 67.3$, which is high, but not disturbingly so.