Zeros of $J_n(\lambda)Y_n(\eta\lambda) - J_n(\eta\lambda)Y_n(\lambda)$*

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The roots, λ_s, $s = 1, 2, \cdots$, of the Bessel function equation

$$J_n(\lambda)Y_n(\eta\lambda) - J_n(\eta\lambda)Y_n(\lambda) = 0$$

are necessary for the solution of a variety of physical problems dealing with material occupying the annular region between two concentric cylinders. η is the ratio of the radii of the two cylinders ($\eta < 1$). Published values of the roots of this equation exist for only scattered values of η for small values of n, and except for those of Chandrasekhar and Elbert [1] are not accurate to more than a few digits. The roots of this equation were first calculated using the McMahon [4] asymptotic expansion.

Since this method is not accurate for small values of η, or for the higher values of n for s small, the solutions were checked by a direct evaluation of the Bessel functions (on a 7094 computer, using a standard Bessel function subroutine). The Bessel functions were computed for successive values of λ, for a given η, until the cross product changed signs. The sign change was then pinpointed between values of the argument, differing from each other by no more than 0.00001. When the roots extracted by this method were within 0.00002 of those obtained by the McMahon solution, then that solution was used for the rest of the roots for that value n.

This subroutine broke down abruptly for arguments greater than 50, and a different subroutine was employed for the larger arguments. A check of the solutions obtained by the two subroutines for arguments in the range of 40 to 50 yielded exactly the same results. The roots for the η's greater than zero were checked against all of the published values which we could locate [3]. The results differed from those of Chandrasekhar and Elbert and Fettis and Caslin [2], by no more than ± 4 in the fifth decimal place. The table of the roots of $\eta = 0$ was taken directly from the Royal Society tables [5]. The roots of this equation were computed for $\eta = 0(0.05)0.95$, $n = 0(1)10$, and $s = 1(1)10$. In the present paper only the roots for $\eta = 0(0.1)0.9$ for $n = 0(1)10$ and $s = 1(1)10$ are given. We have also computed the zeros of $J_n'(\lambda)Y_n'(\lambda\eta) - J_n'(\lambda\eta)Y_n'(\lambda)$ for $\eta = 0(0.05)0.95$, $n = 0(1)10$ and $s = 1(1)10$. These tables are deposited with the Unpublished Mathematical Tables file.

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Received November 28, 1966. Revised March 1, 1967.

* Research supported by the Atmospheric Sciences Section, National Science Foundation, NSF Grants GP-4355 and GP-471.

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Roots of $J_n(\lambda)Y_n(\eta\lambda) - J_n(\eta\lambda)Y_n(\lambda) = 0$

$\eta = 0.0$

s \ n	0	1	2	3	4	5	6	7	8	9	10
1	2.404825	3.831706	5.135622	6.380161	7.588342	8.771483	9.936109	11.08637	12.22509	13.35430	14.47550
2	5.520078	7.015586	8.417244	9.761023	11.06470	12.33860	13.58929	14.82126	16.03777	17.24122	18.43346
3	8.653727	10.17346	11.61984	13.01520	14.37253	15.70017	17.00382	18.28758	19.55453	20.80704	22.04698
4	11.79153	13.32369	14.79595	16.22346	17.61596	18.98013	20.32078	21.64154	22.94517	24.23388	25.50945
5	14.93091	16.47063	17.95981	19.40941	20.82693	22.21780	23.58608	24.93492	26.26681	27.58374	28.88737
6	18.07106	19.61585	21.11699	22.58273	24.01902	25.43034	26.82015	28.19118	29.54566	30.88537	32.21185
7	21.21163	22.76008	24.27011	25.74816	27.19908	28.62661	30.03372	31.42279	32.79580	34.15437	35.49990
8	24.35247	25.90367	27.42057	28.90835	30.37100	31.81171	33.23304	34.63708	36.02561	37.40010	38.76180
9	27.49347	29.04682	30.56920	32.06485	33.53713	34.98878	36.42202	37.83871	39.24044	40.62855	42.00419
10	30.63460	32.18968	33.71652	35.21867	36.69900	38.15986	39.60323	41.03077	42.44388	43.84380	45.23157

$\eta = 0.1$

s \ n	0	1	2	3	4	5	6	7	8	9	10
1	3.313942	3.940940	5.142341	6.380455	7.588356	8.771476	9.936103	11.08637	12.22509	13.35430	14.47550
2	6.857582	7.330574	8.457405	9.764106	11.06488	12.33861	13.58928	14.82126	16.03777	17.24121	18.43346
3	10.37741	10.74837	11.73854	13.02979	14.37375	15.70024	17.00382	18.28759	19.55454	20.80704	22.04698
4	13.88642	14.18863	15.04407	16.26813	17.62120	18.98058	20.32082	21.64154	22.94516	24.23387	25.50945
5	17.38962	17.64330	18.38338	19.51281	20.84345	22.21967	23.58625	24.93494	26.26680	27.58374	28.88736
6	20.88939	21.10730	21.75310	22.77988	24.06032	25.43634	26.82082	28.19124	29.54566	30.88538	32.21186
7	24.38694	24.57756	25.14716	26.07535	27.28562	28.64254	30.03588	31.42302	32.79581	34.15438	35.49991
8	27.88297	28.05219	28.55995	29.39925	30.52886	31.84771	33.23902	34.63785	36.02570	37.40010	38.76180
9	31.37795	31.52994	31.98703	32.74863	33.79537	35.06020	36.43636	37.84091	39.24071	40.62857	42.00418
10	34.87213	35.01001	35.42507	36.11965	37.08683	38.28687	39.63378	41.03630	42.44469	43.84389	45.23159

$\eta = 0.2$

s \ n	0	1	2	3	4	5	6	7	8	9	10
1	3.815956	4.235740	5.221763	6.394599	7.590370	8.771736	9.936134	11.08637	12.22509	13.35430	14.47550
2	7.785530	8.055351	8.803947	9.873893	11.09079	12.34359	13.59012	14.82138	16.03778	17.24121	18.43346
3	11.73210	11.92658	12.49359	13.38065	14.49678	15.73444	17.01169	18.28916	19.55481	20.80709	22.04698
4	15.67015	15.82103	16.26828	16.99396	17.96138	19.10910	20.36064	21.65193	22.94753	24.23436	25.50954
5	19.60421	19.72705	20.09351	20.69683	21.52297	22.54479	23.71630	24.97856	26.27928	27.58687	28.88808
6	23.53607	23.63948	23.94886	24.46134	25.17142	26.06818	27.13018	28.32065	29.59177	30.89952	32.21567
7	27.46662	27.55585	27.82306	28.26702	28.88539	29.67389	30.62377	31.71715	32.92331	34.20201	35.51538
8	31.39631	31.47472	31.70970	32.10067	32.64656	33.34574	34.19484	35.18654	36.30541	37.52491	38.81026
9	35.32543	35.39533	35.60492	35.95388	36.44175	37.06786	37.83109	38.72881	39.75461	40.89479	42.12583
10	39.25414	39.31719	39.50626	39.82122	40.26183	40.82785	41.51902	42.33460	43.27245	44.32677	45.48513

Roots of $J_n(\lambda)Y_n(\eta\lambda) - J_n(\eta\lambda)Y_n(\lambda) = 0$

$\eta = 0.3$

s \ n	0	1	2	3	4	5	6	7	8	9	10
1	4.412391	4.705772	5.470237	6.493721	7.622840	8.781013	9.938545	11.08696	12.22522	13.35433	14.47550
2	8.932838	9.104240	9.600273	10.37131	11.34801	12.45633	13.63324	14.83618	16.04244	17.24259	18.43384
3	13.43413	13.55316	13.90544	14.47701	15.24549	16.18064	17.24495	18.39736	19.59995	20.82427	22.05303
4	17.92925	18.01998	18.29035	18.73508	19.34563	20.11035	21.01384	22.03603	23.15178	24.33304	25.55312
5	22.42163	22.49480	22.71344	23.07505	23.57563	24.20981	24.97087	25.85019	26.83644	27.91449	29.06526
6	26.91261	26.97386	27.15712	27.46103	27.88340	28.42131	29.07126	29.82906	30.68953	31.64600	32.68932
7	31.40276	31.45540	31.61302	31.87482	32.23944	32.70513	33.26976	33.93093	34.68586	35.53133	36.46321
8	35.89237	35.93851	36.07675	36.30654	36.62703	37.03704	37.53516	38.11983	38.78934	39.54180	40.37505
9	40.38163	40.42270	40.54577	40.75047	41.03620	41.40215	41.84734	42.37065	42.97092	43.64685	44.39714
10	44.87064	44.90763	45.01852	45.20303	45.46073	45.79103	46.19323	46.66649	47.20995	47.82267	48.50370

$\eta = 0.4$

s \ n	0	1	2	3	4	5	6	7	8	9	10
1	5.183067	5.391181	5.965934	6.799644	7.789604	8.863166	9.975837	11.10280	12.23162	13.35680	14.47643
2	10.44324	10.55773	10.89443	11.43471	12.15163	13.01393	13.98920	15.04639	16.15816	17.30245	18.46317
3	15.68842	15.76645	15.99866	16.37960	16.90084	17.55167	18.31977	19.19149	20.15207	21.18584	22.27663
4	20.92918	20.98819	21.16448	21.45575	21.85849	22.36805	22.97388	23.68488	24.47936	25.35506	26.30405
5	26.16808	26.21547	26.35729	26.59240	26.91895	27.33454	27.83623	28.42075	29.08448	29.82360	30.63462
6	31.40602	31.44561	31.56415	31.76099	32.03508	32.38501	32.80901	33.30512	33.87113	34.50469	35.20334
7	36.64341	36.67739	36.77918	36.94839	37.18433	37.48613	37.85263	38.28258	38.77451	39.32683	39.93793
8	41.88046	41.91021	41.99939	42.14771	42.35474	42.61986	42.94232	43.32122	43.75556	44.24424	44.78607
9	47.11727	47.14373	47.22306	47.35507	47.53945	47.77575	48.06346	48.40192	48.79041	49.22814	49.71422
10	52.35391	52.37774	52.44918	52.56810	52.73426	52.94735	53.20698	53.51267	53.86388	54.26001	54.70043

$\eta = 0.5$

s \ n	0	1	2	3	4	5	6	7	8	9	10
1	6.246055	6.393150	6.813835	7.457740	8.266731	9.190040	10.18892	11.23571	12.31130	13.40296	14.50237
2	12.54686	12.62470	12.85553	13.23185	13.74233	14.37329	15.10996	15.93740	16.84113	17.80754	18.82404
3	18.83641	18.88892	19.04570	19.30449	19.66172	20.11278	20.65224	21.27417	21.97237	22.74050	23.57222
4	25.12284	25.16240	25.28076	25.47696	25.74948	26.09627	26.51483	27.00234	27.55571	28.17170	28.84694
5	31.40799	31.43970	31.53468	31.69243	31.91216	32.19281	32.53304	32.93129	33.38584	33.89481	34.45624
6	37.69249	37.71895	37.79822	37.93004	38.11394	38.34928	38.63529	38.97101	39.35540	39.78730	40.26545
7	43.97662	43.99932	44.06733	44.18050	44.33853	44.54102	44.78746	45.07726	45.40970	45.78406	46.19947
8	50.26051	50.28038	50.33993	50.43906	50.57757	50.75518	50.97156	51.22629	51.51891	51.84888	52.21561
9	56.54424	56.56192	56.61487	56.70305	56.82631	56.98445	57.17724	57.40439	57.66556	57.96036	58.28838
10	62.82787	62.84378	62.89145	62.97085	63.08188	63.22438	63.39819	63.60308	63.83882	64.10512	64.40166

Roots of $J_n(\lambda)Y_n(\eta\lambda) - J_n(\eta\lambda)Y_n(\lambda) = 0$

$\eta = 0.6$

s \ n	0	1	2	3	4	5	6	7	8	9	10
1	7.828437	7.930091	8.227165	8.698722	9.316810	10.05255	10.87995	11.77729	12.72730	13.71651	14.73455
2	15.69483	15.74727	15.90363	16.16098	16.51482	16.95938	17.48802	18.09368	18.76914	19.50737	20.30161
3	23.55314	23.58832	23.69359	23.86806	24.11035	24.41857	24.79044	25.22336	25.71448	26.26079	26.85921
4	31.40931	31.43575	31.51498	31.64660	31.83004	32.06447	32.34886	32.68203	33.06261	33.48915	33.96006
5	39.26460	39.28578	39.34926	39.45486	39.60225	39.79102	40.02062	40.29042	40.59970	40.94763	41.33333
6	47.11947	47.13713	47.19007	47.27819	47.40131	47.55918	47.75148	47.97784	48.23781	48.53090	48.85659
7	54.97408	54.98923	55.03463	55.11023	55.21591	55.35151	55.51684	55.71166	55.93567	56.18857	56.46999
8	62.82853	62.84178	62.88153	62.94772	63.04027	63.15910	63.30405	63.47496	63.67165	63.89390	64.14146
9	70.68288	70.69466	70.73000	70.78885	70.87118	70.97690	71.10592	71.25812	71.43337	71.63152	71.85238
10	78.53715	78.54776	78.57957	78.63255	78.70668	78.80190	78.91813	79.05529	79.21328	79.39199	79.59130

$\eta = 0.7$

s \ n	0	1	2	3	4	5	6	7	8	9	10
1	10.45523	10.52202	10.71987	11.04154	11.47640	12.01180	12.63456	13.33197	14.09241	14.90569	15.76299
2	20.93546	20.96938	21.07082	21.23885	21.47191	21.76799	22.12462	22.53899	23.00808	23.52869	24.09758
3	31.41025	31.43293	31.50088	31.61381	31.77127	31.97262	32.21706	32.50367	32.83135	33.19898	33.60526
4	41.88363	41.90067	41.95171	42.03666	42.15530	42.30738	42.49254	42.71038	42.96041	43.24208	43.55484
5	52.35646	52.37009	52.41096	52.47901	52.57413	52.69619	52.84501	53.02037	53.22202	53.44968	53.70301
6	62.82900	62.84037	62.87443	62.93118	63.01055	63.11244	63.23677	63.38341	63.55220	63.74297	63.95555
7	73.30138	73.31112	73.34033	73.38899	73.45706	73.54450	73.65123	73.77718	73.92225	74.08633	74.26930
8	83.77366	83.78218	83.80775	83.85034	83.90993	83.98649	84.07997	84.19032	84.31748	84.46137	84.62191
9	94.24587	94.25344	94.27617	94.31403	94.36702	94.43511	94.51826	94.61644	94.72961	94.85771	95.00068
10	104.7180	104.7248	104.7453	104.7794	104.8271	104.8883	104.9632	105.0516	105.1536	105.2690	105.3978

Roots of $J_n(\lambda)Y_n(\eta\lambda) - J_n(\eta\lambda)Y_n(\lambda) = 0$

$\eta = 0.8$

s \ n	0	1	2	3	4	5	6	7	8	9	10
1	15.69808	15.73755	15.85535	16.04973	16.31797	16.65642	17.06089	17.52674	18.04918	18.62336	19.24460
2	31.41095	31.43080	31.49029	31.58918	31.72712	31.90361	32.11801	32.36959	32.65747	32.98073	33.33837
3	47.12056	47.13382	47.17354	47.23969	47.33213	47.45073	47.59529	47.76558	47.96134	48.18222	48.42794
4	62.82935	62.83931	62.86911	62.91877	62.98824	63.07744	63.18629	63.31470	63.46254	63.62969	63.81600
5	78.53780	78.54578	78.56962	78.60937	78.66499	78.73644	78.82369	78.92668	79.04534	79.17962	79.32943
6	94.24609	94.25274	94.27261	94.30575	94.35212	94.41171	94.48448	94.57043	94.66949	94.78164	94.90683
7	109.9543	109.9600	109.9770	110.0054	110.0451	110.0962	110.1587	110.2324	110.3174	110.4137	110.5211
8	125.6624	125.6674	125.6823	125.7071	125.7419	125.7867	125.8413	125.9058	125.9803	126.0646	126.1587
9	141.3705	141.3749	141.3882	141.4103	141.4412	141.4810	141.5295	141.5869	141.6531	141.7281	141.8119
10	157.0786	157.0826	157.0945	157.1144	157.1422	157.1780	157.2217	157.2734	157.3330	157.4005	157.4759

$\eta = 0.9$

s \ n	0	1	2	3	4	5	6	7	8	9	10
1	31.41150	31.42915	31.48206	31.57001	31.69275	31.84986	32.04082	32.26506	32.52187	32.81047	33.13005
2	62.82961	62.83845	62.86495	62.90911	62.97087	63.05019	63.14700	63.26122	63.39276	63.54154	63.70737
3	94.24626	94.25215	94.26983	94.29929	94.34051	94.39348	94.45819	94.53461	94.62270	94.72248	94.83383
4	125.6625	125.6669	125.6802	125.7023	125.7332	125.7730	125.8215	125.8789	125.9451	126.0201	126.1038
5	157.0786	157.0822	157.0928	157.1105	157.1352	157.1670	157.2059	157.2518	157.3048	157.3648	157.4319
6	188.4947	188.4976	188.5065	188.5212	188.5418	188.5634	188.6008	188.6390	188.6832	188.7333	188.7892
7	219.9107	219.9132	219.9208	219.9334	219.9511	219.9739	220.0016	220.0345	220.0723	220.1152	220.1632
8	251.3267	251.3289	251.3355	251.3466	251.3621	251.3820	251.4063	251.4350	251.4681	251.5057	251.5476
9	282.7427	282.7446	282.7505	282.7604	282.7741	282.7918	282.8134	282.8389	282.8684	282.9018	282.9391
10	314.1586	314.1604	314.1657	314.1745	314.1869	314.2028	314.2223	314.2453	314.2718	314.3018	314.3354

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More on the Calculation of the Integral

$$I_n(b) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos bx\, dx$$

By Henry E. Fettis

The evaluation of this integral has been the subject of two recent papers [1], [2]. Although the integral can be expressed in a simple analytical form, namely

$$(1)\qquad \left\{ I_n(b) = \frac{n}{2^{n-1}} \sum_{k=0}^{[(n-b)/2\}} \frac{(-)^k(n-b-2k)^{n-1}}{k!(n-k)!}, b < n \right\}, \quad = 0, \quad b \geqq n,$$

(where $[(n-b)/2\}$ denotes the largest integer less than $(n-b)/2$), the use of the above expression for large n has not proved satisfactory. Alternative schemes in lieu of (1) have been proposed by Medhurst and Roberts [1] and Thompson [2]. These essentially are recursive-type methods, in which results for higher values of n and b are computed from starting values obtained for lower order and argument by the exact expression (1). Such schemes have the disadvantage that the direct computation for a given n and b is not possible. The present paper proposes a method which overcomes this difficulty and allows the integral to be computed directly. The formulae work equally well for small and large values of b, and are particularly well suited to computation for moderate and large n.

The basis of the present method is the Poisson summation formula [3]. In its most general form it may be written as follows

$$(2)\quad \sum_{k=-\infty}^{\infty} \exp\,[iku_1]f(a+kd) = \frac{1}{d}\sum_{m=-\infty}^{\infty} G\left(\frac{2\pi m + u}{d}\right)\exp\,[-i(a/d)(2\pi m + u_1)]$$

where G is the Fourier transform of f, namely

Received May 18, 1967.