On Numerical Calculation
of Transonic Flow Patterns*

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1. Introduction. Obtaining a mathematical expression for the solution of a problem is frequently not the chief difficulty in developing a practical numerical procedure. This is often the case when the expression involves infinite processes such as integration and differentiation. It then becomes necessary to approximate these infinite processes by some finite process which can be carried out with the use of a computer.

The purpose of the present paper is to describe a numerical procedure for computing families of transonic flow patterns for compressible fluids. The basis for this procedure is provided by a family of particular solutions of the Chaplygin equation (2.1). These solutions were obtained independently by Bergman [1], [2] and by Bers and Gelbart [6], [7].

Although these solutions have a fairly simple mathematical form, they involve multiple integrals of complicated functions which must be approximated in some way in order to make their numerical computation feasible. This difficulty is overcome by approximating the functions involved by polynomials and then integrating these polynomials.

When applying "the method of particular solutions" to solve boundary-value problems, we form a linear combination

$$\Psi^{(N)} = \sum_{\nu=1}^{N} A_\nu \psi_\nu$$

of particular solutions ψ_ν and determine the coefficients A_ν so that $\Psi^{(N)}$ approximates the prescribed boundary values. Thus, the computation of a set of particular solutions is of interest not only in itself but also as a first step in the solution of certain boundary-value problems.

2. The Partial Differential Equations Arising in the Theory of Compressible Fluid Flow. It is well known that the partial differential equation satisfied by the stream function ψ of a two-dimensional, inviscid, irrotational, compressible fluid flow when considered in the physical plane (i.e., when ψ is considered as a function of the coordinates x, y of the plane in which the flow takes place) is nonlinear. However, if we consider the equation in the hodograph plane, i.e., if we consider ψ as a function of the speed v and the angle θ which the velocity vector makes with a fixed direction (say, with the x-axis), then, as Chaplygin [8] and Molenbroek [11] have shown, ψ satisfies the linear partial differential equation

$$(2.1) \qquad \frac{1 - M^2}{\rho^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{v}{\rho} \frac{\partial}{\partial v} \left(\frac{v}{\rho} \frac{\partial \psi}{\partial v} \right) = 0 .$$

Here ρ denotes the density which is related to the pressure p by the relation $p = c\rho^k$, where c and k are constants ($k = 1.4$ for air). The Mach number M and the density ρ are related to the speed v by

$$(2.2) \qquad \rho = (1 - (k - 1)v^2/2)^{1/(k-1)} ,$$

$$(2.3) \qquad M = \frac{v}{(1 - (k - 1)v^2/2)^{1/2}} .$$

Since the denominator in (2.3) is the local speed of sound, M will be smaller than one if the flow is subsonic and larger than one if it is supersonic. Instead of the speed v we may introduce the variable

$$(2.4) \qquad H = \int_{v_1}^{v} \frac{\rho}{v} \, dv = \int_{v_1}^{v} \left(1 - \frac{k - 1}{2} v^2\right)^{1/(k-1)} \frac{dv}{v} ,$$

where v_1 denotes the velocity corresponding to $M = 1$. If one introduces the potential ϕ as well as the stream function ψ, then the differential equations connecting these quantities assume the form

$$(2.5) \qquad \frac{\partial\phi}{\partial\theta} = \frac{\partial\psi}{\partial H} , \qquad \frac{\partial\phi}{\partial H} = - l(H) \frac{\partial\psi}{\partial\theta} , \qquad l(H) = \frac{1 - M^2}{\rho^2} .$$

(See [3], [10].)

From (2.5) we deduce at once that the differential equation for ψ may be written in the form

$$(2.6) \qquad \frac{\partial^2\psi}{\partial H^2} + l(H) \frac{\partial^2\psi}{\partial\theta^2} = 0 .$$

Equation (2.6) is a differential equation of mixed type, since $l(H) > 0$ for $H < 0$ (i.e., for $v < v_1$, $M < 1$), while $l(H) < 0$ for $H > 0$ (i.e., for $v > v_1$, $M > 1$).

In order to transform a solution of (2.6) from the (H, θ)-plane to the physical plane, we use the relations

$$(2.7) \qquad
\begin{aligned}
x &= \int \left[\frac{(M^2 - 1)\cos\theta}{\rho v^2} \psi_\theta - \frac{\sin\theta}{\rho v} \psi_v\right]dv + \left[\frac{\cos\theta}{\rho} \psi_v - \frac{\sin\theta}{\rho v} \psi_\theta\right]d\theta , \\
y &= \int \left[\frac{(M^2 - 1)\sin\theta}{\rho v^2} \psi_\theta + \frac{\cos\theta}{\rho v} \psi_v\right]dv + \left[\frac{\sin\theta}{\rho} \psi_v + \frac{\cos\theta}{\rho v} \psi_\theta\right]d\theta .
\end{aligned}$$

If the integration is along a streamline, $\psi_v dv + \psi_\theta d\theta = 0$ and (2.7) becomes

$$(2.8) \qquad
\begin{aligned}
x &= \int \frac{\cos\theta}{\rho} \left[\frac{M^2 - 1}{v^2} \psi_\theta dv + \psi_v d\theta\right] , \\
y &= \int \frac{\sin\theta}{\rho} \left[\frac{M^2 - 1}{v^2} \psi_\theta dv + \psi_v d\theta\right] .
\end{aligned}$$

Note that if the Jacobian of this transformation vanishes at a point, the transformation can fail to be one-to-one in a neighborhood of the point, and the flow may have no physical significance in the neighborhood. See, e.g., [5, p. 17] or [10, p. 311 ff.].

3. The Particular Solutions and Their Approximation. The particular solutions of (2.5) and (2.6), with which we are concerned, are a generalization of the harmonic polynomials obtained by taking real and imaginary parts of the analytic functions $(\theta + iH)^n$. These solutions $\phi_{n,j}(H, \theta)$ and $\psi_{n,j}(H, \theta)$, $j = 1, 2$, $n = 0, 1, \cdots$, may be written in the form

$$\phi_{n,1}(H,\theta) + i\psi_{n,1}(H,\vartheta) = (\theta + iH)^{[n]}$$

(3.1)
$$\equiv \left[\theta^n - 2!\binom{n}{2}\theta^{n-2}p_2(H) + 4!\binom{n}{4}\theta^{n-4}p_4(H) + \cdots \right]$$
$$+ i\left[1!\binom{n}{1}\theta^{n-1}s_1(H) - 3!\binom{n}{3}\theta^{n-3}s_3(H) + \cdots \right],$$

$$\phi_{n,2}(H,\theta) + i\psi_{n,2}(H,\theta) = i \odot (\theta + iH)^{[n]}$$

(3.2)
$$\equiv - \left[1!\binom{n}{1}\theta^{n-1}p_1(H) - 3!\binom{n}{3}\theta^{n-3}p_3(H) + \cdots \right]$$
$$+ i\left[\theta^n - 2!\binom{n}{2}\theta^{n-2}s_2(H) + 4!\binom{n}{4}\theta^{n-4}s_4(H) - \cdots \right]$$

where

(3.3)
$$p_0(H) = 1, \qquad p_1(H) = \int_{H_0}^{H} l(H_1)dH_1,$$
$$p_m(H) = \int_{H_0}^{H} l(H_m) \int_{H_0}^{H_m} p_{m-2}(H_{m-1})dH_{m-1}dH_m$$

and

(3.4)
$$s_m(H) = \int_{H_0}^{H} \int_{H_0}^{H_m} l(H_{m-1})s_{m-2}(H_{m-1})dH_{m-1}dH_m,$$
$$s_0(H) = 1, \qquad s_1(H) = H - H_0 = \int_{H_0}^{H} dH_1.$$

(See [2], [3] and [6].) It will be convenient to choose $H_0 = 0$ corresponding to $M = 1$.

Because of the nature of the function $l(H)$, the integrals involved in the calculation of $s_m(H)$ would be very difficult to evaluate exactly. Therefore, we approximate $l(H)$ in suitable intervals by polynomials and then integrate these polynomials to obtain polynomial approximations for the functions $s_m(H)$. The intervals are

$$J_1: -1.0 \leqq H \leqq 0 \quad \text{corresponding to} \quad .284 \leqq M \leqq 1,$$

and

$$J_2: 0 \leqq H \leqq 0.2 \quad \text{corresponding to} \quad 1 \leqq M \leqq 1.78.$$

The polynomial approximation of $l(H)$ in J_κ is denoted by

(3.5)
$$\tilde{l}_\kappa(H) = \sum_{\nu=0}^{10} A_\nu^{(\kappa)}\eta_\kappa^\nu, \qquad \kappa = 1, 2, \qquad \eta_1 = H, \qquad \eta_2 = 5H.$$

In order to determine the coefficients of the approximating polynomial in these intervals, we need an algorithm for the calculation of $l(H)$ as a function of H. We consider the special case of air, for which $k = 1.4$. Since ρ and M are functions of v, as shown by Eqs. (2.2) and (2.3), $l(H)$ can be expressed as a function of v using its definition in (2.5). In this special case the integral in (2.4) can be evaluated in closed form and yields

$$
\text{(3.6)}\qquad
\begin{aligned}
H = \Bigg\{ & (1 - 0.2v^2)^{1/2}\left[\frac{(1 - 0.2v^2)^2}{5} + \frac{1 - 0.2v^2}{3} + 1\right] \\
& - \tfrac{1}{2}\ln\frac{1 + (1 - 0.2v^2)^{1/2}}{1 - (1 - 0.2v^2)^{1/2}} \Bigg\}\Bigg|_{v_1}^{v}
\end{aligned}
$$

where $v_1 = (5/6)^{1/2}$. We use Newton's method to calculate v corresponding to any value of H in J_1 or J_2 and then $l(H)$ can be calculated at once.

The Remez algorithm [12], as adapted for the Burroughs B5500 computer by Golub and Smith [9], was used to determine polynomials $\tilde{l}_\kappa(H)$ which are the best approximations to $l(H)$ in the Chebyshev sense on the intervals J_κ. The values of the coefficients $A_\nu^{(\kappa)}$, $\kappa = 1, 2$, $\nu = 0, 1, \cdots, 10$, are given in Table 1. From the approximation algorithm we know that

TABLE 1

Values of the coefficients $A_\nu^{(\kappa)}$ of the approximations to $l(H)$, see (3.5)

ν \ κ	1	2
0	0.0001711222	—0.0025985374
1	−9.3681945493	−1.3189899758
2	−55.284916968	−22.598654078
3	−248.40627057	280.04321217
4	−840.40978809	−2002.2845890
5	−2052.0104328	8142.5278038
6	−3488.0712341	−20236.235647
7	−3982.8542755	31196.920595
8	−2896.2958023	−29181.568535
9	−1207.8174204	15188.597028
10	−219.39926509	−3388.9118171

$$
\text{(3.7)}\qquad
\begin{aligned}
\max_{J_1}|l(H) - \tilde{l}_1(H)| &\leq 1.7 \times 10^{-4} \\
\max_{J_2}|l(H) - \tilde{l}_2(H)| &\leq 2.6 \times 10^{-3}.
\end{aligned}
$$

The function $l(H)$ becomes very large and negative when H exceeds 0.2. Indeed, as $H \to 0.2513$, $M \to \infty$, $v \to \sqrt{5}$ and $l(H) \to -\infty$. It was therefore practically impossible to obtain a good polynomial approximation for $H > 0.2$.

The polynomial approximations for the $s_m(H)$ in the interval J_κ, denoted by $\tilde{s}_m^{(\kappa)}(H)$, satisfy the recurrence relation given by (3.4) and can be expressed as

$$(3.8) \quad \tilde{s}_2^{(\kappa)}(H) = H^2 \sum_{\nu=0}^{10} C_\nu^{(2,\kappa)} \eta_\kappa^\nu, \qquad C_\nu^{(2,\kappa)} = \frac{A_\nu^{(\kappa)}}{(\nu+1)(\nu+2)}, \qquad \kappa = 1, 2,$$

$$(3.9) \quad \tilde{s}_3^{(\kappa)}(H) = H^3 \sum_{\nu=0}^{10} C_\nu^{(3,\kappa)} \eta_\kappa^\nu, \qquad C_\nu^{(3,\kappa)} = \frac{A_\nu^{(\kappa)}}{(\nu+2)(\nu+3)}, \qquad \kappa = 1, 2,$$

and, in general,

$$(3.10) \qquad \tilde{s}_m^{(\kappa)}(H) = H^m \sum_{\nu=0}^{10[m/2]} C_\nu^{(m,\kappa)} \eta_\kappa^\nu \qquad \kappa = 1, 2,$$

where

$$(3.11) \qquad C_\nu^{(m,\kappa)} = \sum_{\{\xi \,|\, 0 \leq \xi \leq 10, 0 \leq \nu - \xi \leq 10[(m-2)/2]\}} \frac{A_\xi^{(\kappa)} C_{\nu-\xi}^{(m-2,\kappa)}}{(\nu+m-1)(\nu+m)}.$$

(Note that $[m/2] = m/2$ if m is even but $[m/2] = (m-1)/2$ if m is odd.) Letting $\tilde{s}_m(H) = \tilde{s}_m^{(\kappa)}(H)$ for H in J_κ, the approximations to the stream functions $\psi_{n,1}$ and $\psi_{n,2}$, denoted by $\tilde{\psi}_{n,1}$ and $\tilde{\psi}_{n,2}$, are calculated using Eqs. (3.1) and (3.2) with $s_m(H)$ replaced by $\tilde{s}_m(H)$. It is clear that the approximations satisfy

$$(3.12) \qquad \tilde{\psi}_{HH} + \tilde{l}(H)\tilde{\psi}_{\theta\theta} = 0, \qquad \tilde{l}(H) = \tilde{l}_\kappa(H) \quad \text{for} \quad H \quad \text{in} \quad J_\kappa.$$

4. Computational Results. It is seen from Table 1 that the coefficients of the polynomials $\tilde{l}_\kappa(H)$ are very large; they are of constant sign in $\tilde{l}_1(H)$, which is used for $H < 0$, and of alternating sign in $\tilde{l}_2(H)$, which is used for $H > 0$. Hence, in both cases the evaluation of these functions and of the functions $\tilde{s}_m^{(\kappa)}(H)$ will involve large terms of alternating sign and this may result in the loss of several significant digits. The magnitude of the error given in (3.7) indicates that single-precision arithmetic (about 11 decimal digits) on the Stanford Burroughs B5500 computer is adequate for the calculation of $\tilde{l}_\kappa(H)$. However, *for the evaluation of $\tilde{s}_m^{(\kappa)}(H)$ and the subsequent calculation of $\tilde{\psi}_{n,j}$, single-precision arithmetic was found to be inadequate.* Hence these calculations were carried out in double-precision arithmetic (78-bit arithmetic equivalent to about 22 decimal digits) on the B5500. Some typical values of $\tilde{\psi}_{n,j}$ are given in Table 2.

As a measure of how well the approximate solutions satisfy the differential equation (2.6), we introduce the quantity

$$(4.1) \qquad M_G(\tilde{\psi}) = \left[\frac{\int\int_G |S(\tilde{\psi})|^2 dH d\theta}{\int\int_G |\tilde{\psi}|^2 dH d\theta} \right]^{1/2}$$

where

$$(4.2) \qquad S(\tilde{\psi}) = \partial^2\tilde{\psi}/\partial H^2 + l(H)\, \partial^2\tilde{\psi}/\partial\theta^2.$$

Since $\tilde{\psi}_{n,j}$ satisfies (3.12), we see that

$$(4.3) \qquad S(\tilde{\psi}_{n,j}) = [l(H) - \tilde{l}(H)]\, \partial^2\tilde{\psi}_{n,j}/\partial\theta^2.$$

By formally differentiating Eqs. (3.1) and (3.2), formulas are obtained for $\partial^2\tilde{\psi}_{n,j}/\partial\theta^2$ and these are used to evaluate $S(\tilde{\psi}_{n,j})$.

TABLE 2

Values of $\hat\psi_{n,i}$ and $S(\hat\psi_{n,i})$

$\hat\psi_{3,1}$

$H \rightarrow$	$\theta \rightarrow$ 0.50		1.00		2.00	
	$\hat\psi$	$S(\hat\psi)$	$\hat\psi$	$S(\hat\psi)$	$\hat\psi$	$S(\hat\psi)$
−1.00	0.1633700	−0.0010265	−2.0866300	−0.0010265	−11.0866300	−0.0010265
−0.90	−0.0196152	−0.0008397	−2.0446152	−0.0008397	−10.1446152	−0.0008397
−0.80	−0.1489456	−0.0007931	−1.9489456	−0.0007931	−9.1489456	−0.0007931
−0.70	−0.2307471	−0.0006517	−1.8057471	−0.0006517	−8.1057471	−0.0006517
−0.60	−0.2712086	0.0001343	−1.6212086	0.0001343	−7.0212086	0.0001343
−0.50	−0.2765697	0.0003584	−1.4015697	0.0003584	−5.9015697	0.0003584
−0.40	−0.2531522	−0.0004052	−1.1531522	−0.0004052	−4.7531522	−0.0004052
−0.30	−0.2073766	0.0001605	−0.8823766	0.0001605	−3.5823766	0.0001605
−0.20	−0.1457304	0.0000032	−0.5957304	0.0000032	−2.3957304	0.0000032
−0.10	−0.0746577	0.0000084	−0.2996577	0.0000084	−1.1996577	0.0000084
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.02	0.0150008	0.0002828	0.0600008	0.0002828	0.2400008	0.0002828
0.06	0.0450781	0.0008513	0.1800781	0.0008513	0.7200781	0.0008513
0.10	0.0757382	−0.0010744	0.3007382	−0.0010744	1.2007382	−0.0010744
0.14	0.1086399	−0.0011564	0.4236399	−0.0011564	1.6836399	−0.0011564
0.18	0.1488513	−0.0002686	0.5538513	−0.0002686	2.1738513	−0.0002686

$\bar{\psi}_{4,1}$

$H \to$	$\theta \to$ 0.50		1.00		2.00	
	$\bar{\psi}$	$S(\bar{\psi})$	$\bar{\psi}$	$S(\bar{\psi})$	$\bar{\psi}$	$S(\bar{\psi})$
−1.00	1.3267400	−0.0020530	−0.3465201	−0.0041061	−24.6930402	−0.0082121
−0.90	0.8607697	−0.0016795	−0.9784606	−0.0033590	−23.5569213	−0.0067179
−0.80	0.5021087	0.0015863	−1.3957825	0.0031726	−21.9915650	0.0063451
−0.70	0.2385057	−0.0013034	−1.6229885	−0.0026067	−20.0459770	−0.0052135
−0.60	0.0575827	0.0002687	−1.6848346	0.0005373	−17.7696691	0.0010747
−0.50	−0.0531394	0.0007169	−1.6062788	0.0014338	−15.2125576	0.0028676
−0.40	−0.1063044	−0.0008105	−1.4126089	−0.0016210	−12.4252177	−0.0032420
−0.30	−0.1147532	0.0003211	−1.1295064	0.0006422	−9.4590129	0.0012844
−0.20	−0.0914608	0.0000063	−0.7829216	0.0000126	−6.3658431	0.0000253
−0.10	−0.0493153	0.0000168	−0.3986307	0.0000335	−3.1972613	0.0000671
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.02	0.0100016	0.0005656	0.0800033	0.0011312	0.6400065	0.0022624
0.06	0.0301561	0.0017026	0.2403123	0.0034052	1.9206246	0.0068105
0.10	0.0514764	−0.0021488	0.4029527	−0.0042976	3.2059054	−0.0085952
0.14	0.0772797	−0.0023129	0.5745595	−0.0046257	4.5091189	−0.0092515
0.18	0.1177026	−0.0005372	0.7754052	−0.0010745	5.8708104	−0.0021489

TABLE 2. *Continued*

$\bar{\psi}_{8,1}$

$H \downarrow$	$\theta \rightarrow$ 0.50		1.00		2.00	
	$\tilde{\psi}$	$S(\tilde{\psi})$	$\tilde{\psi}$	$S(\tilde{\psi})$	$\tilde{\psi}$	$S(\tilde{\psi})$
−1.00	−1.2978519	−0.0032399	3.3942315	0.0708942	260.5541163	−0.5326284
−0.90	−0.8603520	0.0006907	6.1335915	0.0429059	50.3864492	−0.6400668
−0.80	−0.4549513	−0.0022037	6.1176754	−0.0253736	−119.2155195	0.7808800
−0.70	−0.1746287	0.0021328	4.5614798	0.0084041	−242.3325237	−0.7712069
−0.60	−0.0263269	−0.0003766	2.4492688	0.0006952	−316.8089715	0.1822394
−0.50	0.0243178	−0.0006447	0.4951412	0.0076500	−343.9154953	0.5384575
−0.40	0.0224257	0.0002488	−0.8669683	−0.0141331	−328.0002885	−0.6559222
−0.30	0.0053674	0.0000758	−1.4681934	0.0072598	−276.0623392	0.2736735
−0.20	−0.0056309	0.0000040	−1.3657653	0.0001643	−197.1878526	0.0055587
−0.10	−0.0056587	0.0000140	−0.7808918	0.0004643	−101.7870372	0.0149860
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.02	0.0012514	0.0004952	0.1600456	0.0158389	20.4814581	0.5067926
0.06	0.0038873	0.0015157	0.4843776	0.0478801	61.5799526	1.5271969
0.10	0.0075784	−0.0020668	0.8416296	−0.0616502	103.7251477	−1.9371814
0.14	0.0157638	−0.0027463	1.3289317	−0.0704146	149.9233276	−2.1173111
0.18	0.0430116	−0.0009957	2.2740323	−0.0189875	209.6047726	−0.5123954

$\bar{\psi}_{2,2}$

$\theta \rightarrow$	0.50		1.00		2.00	
$H \downarrow$	$\bar{\psi}$	$S(\bar{\psi})$	$\bar{\psi}$	$S(\bar{\psi})$	$\bar{\psi}$	$S(\bar{\psi})$
−1.00	−0.5216650	0.0003422	0.2283350	0.0003422	3.2283350	0.0003422
−0.90	−0.3588351	0.0003110	0.3911649	0.0003110	3.3911649	0.0003110
−0.80	−0.2158763	−0.0003305	0.5341237	−0.0003305	3.5341237	−0.0003305
−0.70	−0.0927193	0.0003103	0.6572807	0.0003103	3.6572807	0.0003103
−0.60	0.0107555	−0.0000746	0.7607555	−0.0000746	3.7607555	−0.0000746
−0.50	0.0947367	−0.0002390	0.8447367	−0.0002390	3.8447367	−0.0002390
−0.40	0.1595462	0.0003377	0.9095462	0.0003377	3.9095462	0.0003377
−0.30	0.2057550	−0.0001784	0.9557550	−0.0001784	3.9557550	−0.0001784
−0.20	0.2344084	−0.0000053	0.9844084	−0.0000053	3.9844084	−0.0000053
−0.10	0.2475960	−0.0000280	0.9975960	−0.0000280	3.9975960	−0.0000280
0.00	0.2500000	0.0051980	1.0000000	0.0051980	4.0000000	0.0051980
0.02	0.2500265	0.0047133	1.0000265	0.0047133	4.0000265	0.0047133
0.06	0.2508321	0.0047295	1.0008321	0.0047295	4.0008321	0.0047295
0.10	0.2545577	−0.0035813	1.0045577	−0.0035813	4.0045577	−0.0035813
0.14	0.2653657	−0.0027534	1.0153657	−0.0027534	4.0153657	−0.0027534
0.18	0.2928703	−0.0004974	1.0428703	−0.0004974	4.0428703	−0.0004974

$\tilde{\psi}_{4,2}$

$H \rightarrow$	$\theta \rightarrow$ 0.50		1.00		2.00	
	$\tilde{\psi}$	$S(\tilde{\psi})$	$\tilde{\psi}$	$S(\tilde{\psi})$	$\tilde{\psi}$	$S(\tilde{\psi})$
−1.00	−0.4745369	−0.0010710	−3.0095295	0.0004688	−1.8994999	0.0066279
−0.90	−0.4626686	−0.0006696	−2.2649266	0.0007299	1.7760413	0.0063282
−0.80	−0.4078118	−0.0004281	−1.5667551	−0.0010591	5.0474718	−0.0070076
−0.70	−0.3270985	−0.0001726	−0.9318353	0.0012238	7.8992173	0.0068097
−0.60	−0.2352251	−0.0000048	−0.3743253	−0.0003407	10.3192739	−0.0016840
−0.50	−0.1443976	−0.0001358	0.0944175	−0.0012112	12.2996777	−0.0055125
−0.40	−0.0642527	0.0003233	0.4662051	0.0018430	13.8380364	0.0079216
−0.30	−0.0016996	−0.0002202	0.7366981	−0.0010229	14.9402889	−0.0042339
−0.20	0.0393869	−0.0000074	0.9067245	−0.0000311	15.6260751	−0.0001258
−0.10	0.0589006	−0.0000415	0.9855825	−0.0001673	15.9423098	−0.0006705
0.00	0.0625000	0.0077970	1.0000000	0.0311880	16.0000000	0.1247520
0.02	0.0625398	0.0070707	1.0001592	0.0282806	16.0006368	0.1131203
0.06	0.0637490	0.0071178	1.0049936	0.0284005	16.0199718	0.1135312
0.10	0.0693630	−0.0054700	1.0273727	−0.0215860	16.1094116	−0.0860502
0.14	0.0858594	−0.0043840	1.0925050	−0.0167743	16.3690876	−0.0663357
0.18	0.1293510	−0.0008741	1.2597673	−0.0031126	17.0314324	−0.0120663

$\bar{\psi}_{8,2}$

H	$\theta \to$ 0.50		1.00		2.00	
	$\bar{\psi}$	$S(\bar{\psi})$	$\bar{\psi}$	$S(\bar{\psi})$	$\bar{\psi}$	$S(\bar{\psi})$
-1.00	-0.7431922	0.0106611	9.0849024	-0.0170070	-488.1153571	-0.8094140
-0.90	-0.1576505	0.0056536	4.2557871	-0.0223131	-428.3197002	-0.5145511
-0.80	0.0601951	-0.0029789	0.8208215	0.0247473	-335.6181664	0.3165924
-0.70	0.0938495	0.0010005	-1.1527736	-0.0201550	-223.8673185	-0.0941015
-0.60	0.0595074	-0.0000098	-1.8603734	0.0035262	-106.0258446	-0.0213853
-0.50	0.0193905	0.0002468	-1.6511619	0.0063125	6.3884812	-0.1893028
-0.40	-0.0028237	-0.0003460	-0.9328852	-0.0021161	103.8053357	0.4049504
-0.30	-0.0067235	0.0000891	-0.0901534	-0.0018417	179.1288441	-0.2672753
-0.20	-0.0017496	-0.0000003	0.5824978	-0.0001135	228.3664747	-0.0088829
-0.10	0.0028836	-0.0000105	0.9931547	-0.0007546	251.6994570	-0.0496449
0.00	0.0039063	0.0022741	1.0000000	0.1455440	256.0000000	9.3148181
0.02	0.0039179	0.0020654	1.0007431	0.1320253	256.0475523	8.4470969
0.06	0.0042741	0.0021729	1.0233594	0.1340802	257.4921248	8.5016895
0.10	0.0060169	-0.0020053	1.1294681	-0.1071730	264.1970055	-6.5276195
0.14	0.0120356	-0.0024056	1.4521856	-0.0952249	283.8842189	-5.2198640
0.18	0.0349473	-0.0009131	2.3831222	-0.0234227	335.6928553	-1.0368814

These calculations were also carried out in double-precision arithmetic on the B5500. Typical values for $S(\tilde{\psi}_{n,j})$ are given in Table 2 and for $G = \{H, \theta | -1.0 \leq H \leq .2, 0 \leq \theta \leq 2\}$ we have

$$M_G(\tilde{\psi}_{n,1}) \leq 0.01 \quad \text{when } n \leq 7,$$

and

$$M_G(\tilde{\psi}_{n,2}) \leq 0.01 \quad \text{when } n \leq 5.$$

Streamlines for three particular examples,

$$\tilde{\psi}^{(1)} = 5\,\tilde{\psi}_{1,1} + \tilde{\psi}_{5,1} + \tilde{\psi}_{4,2},$$
$$\tilde{\psi}^{(2)} = 5\,\tilde{\psi}_{1,1} + \tilde{\psi}_{4,1} + \tilde{\psi}_{4,2},$$

and

$$\tilde{\psi}^{(3)} = 5\,\tilde{\psi}_{1,1} + \tilde{\psi}_{3,1} + \tilde{\psi}_{3,2},$$
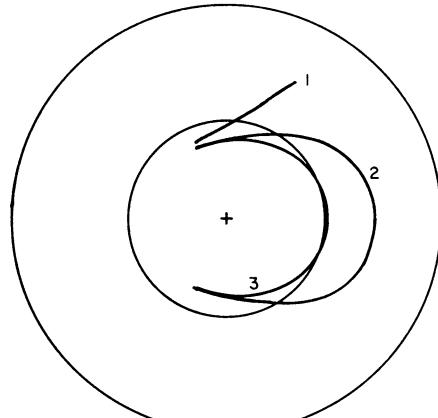
HODOGRAPH PLANE
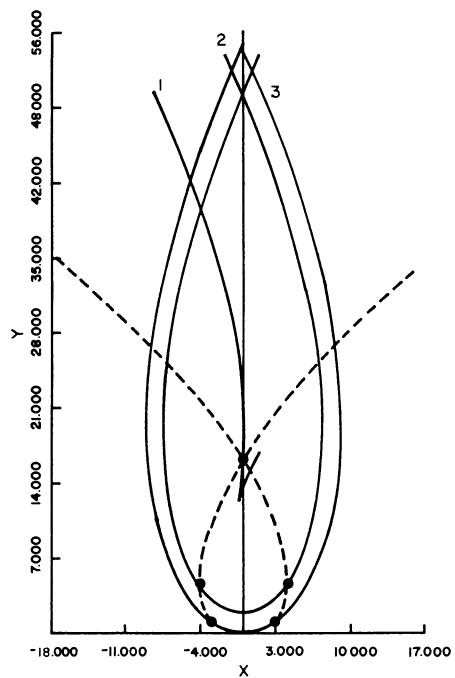
PHYSICAL PLANE

FIGURE 1. $\tilde{\psi}^{(1)} = 5\tilde{\psi}_{1,1} + \tilde{\psi}_{5,1} + \tilde{\psi}_{4,2}$
Streamline 1 $\tilde{\psi}^{(1)} = 5.0625$
Streamline 2 $\tilde{\psi}^{(1)} = 1.0000$
Streamline 3 $\tilde{\psi}^{(1)} = 0.0625$
– – – – –   Sonic Line
●   Transition Points

were computed in the hodograph plane and transformed into the physical plane by the numerical evaluation of the integrals in (2.7) using the trapezoidal rule. The results of these computations are indicated in Figs. 1–3.

As noted above, the transformation from the (H, θ)-plane to the physical plane may fail to be one-to-one. Consequently, the flows represented by the above stream-functions correspond to physically possible flows only in certain subdomains. This explains why some of the streamlines drawn intersect. It should also be observed that not all of the intersections of the streamlines with the sonic line (shown as a dotted line) correspond to transitions between subsonic and supersonic flow. Those intersections which do correspond to such transitions are indicated on the figures.


HODOGRAPH PLANE

PHYSICAL PLANE
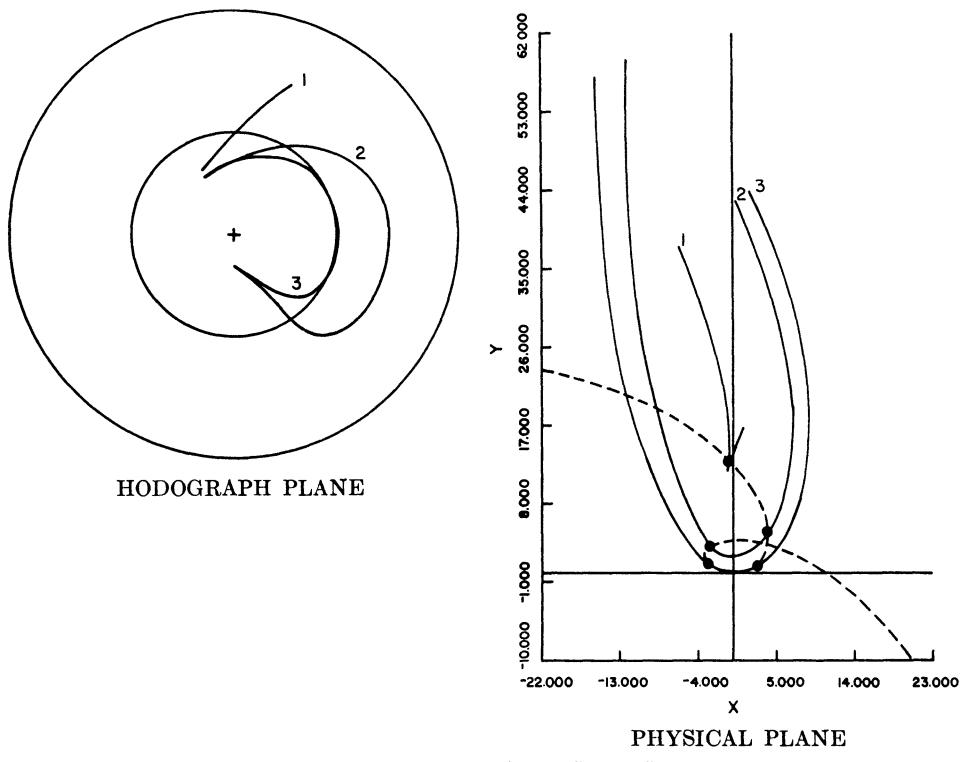
FIGURE 2. $\bar{\psi}^{(2)} = 5\bar{\psi}_{1,1} + \bar{\psi}_{4,1} + \bar{\psi}_{4,2}$
Streamline 1 $\bar{\psi}^{(2)} = 5.0625$
Streamline 2 $\bar{\psi}^{(2)} = 1.0000$
Streamline 3 $\bar{\psi}^{(2)} = 0.0625$
– – – – –   Sonic Line
    •   Transition points

Remark. Stark [13], [14] has used a set of particular solutions generated by integral operators as a basis for another numerical procedure for computing solutions of (2.1). More detailed numerical results from the present procedure can be found in [4].

HODOGRAPH PLANE

PHYSICAL PLANE
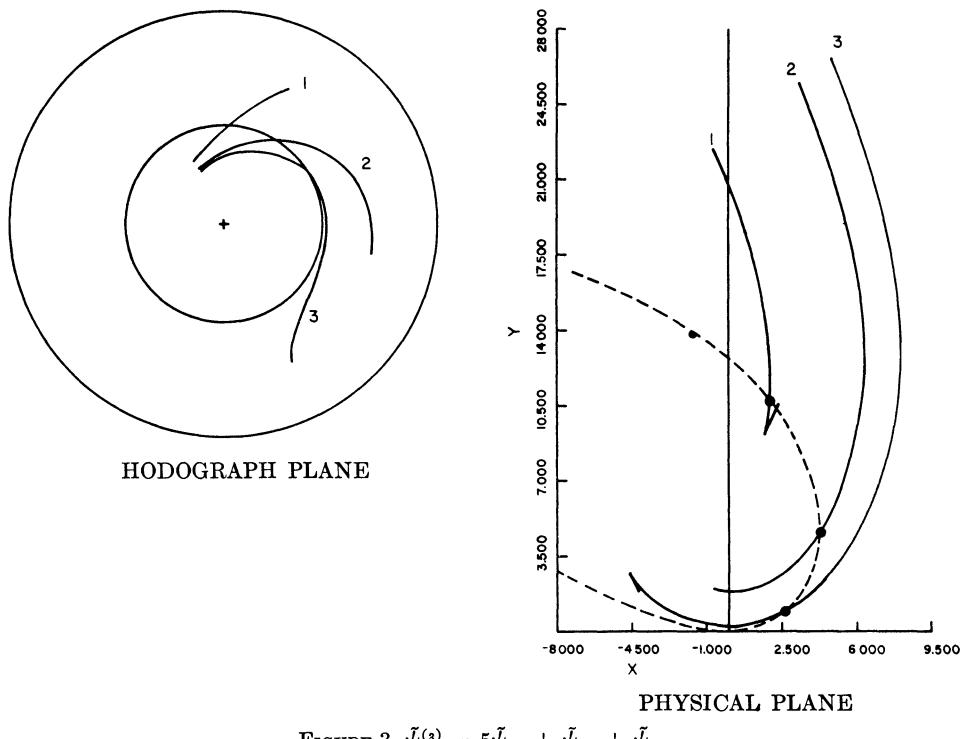
FIGURE 3. $\tilde{\psi}^{(3)} = 5\tilde{\psi}_{1,1} + \tilde{\psi}_{3,1} + \tilde{\psi}_{3,2}$

Streamline 1 $\quad \tilde{\psi}^{(3)} = 3.375$

Streamline 2 $\quad \tilde{\psi}^{(3)} = 1.000$

Streamline 3 $\quad \tilde{\psi}^{(3)} = 0.125$

– – – – – Sonic Line

● Transition Points

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