A Special Case of the Filon Quadrature Formula*

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Introduction. The Filon quadrature formula [1] is used for the numerical evaluation of integrals of the form

$$(1) \qquad S = \int_a^b f(x) \sin (kx)dx\,, \qquad C = \int_a^b f(x) \cos (kx)dx\,.$$

The Filon formula is advantageous over usual numerical integration formulas for smooth $f(x)$, especially for large k, since the number of points which need be tabulated depends on the behavior of $f(x)$ rather than on $f(x) \sin kx$ or $f(x) \cos kx$. Under certain circumstances the Filon quadrature formula reduces to a simple form, namely

$$(2) \qquad S^* = ((-1)^m/k)\{f(a) - f(b)\}\,, \qquad C^* = ((-1)^m/k)\{f(b) - f(a)\}\,,$$

where the asterisk is used to denote the inexact result produced by the quadrature formula; a and the integer m are related through the condition

$$(3) \qquad a = m\pi/k \quad \text{for } S\,, \qquad a = (m + \tfrac{1}{2})\pi/k \quad \text{for } C$$

and, finally, the integration interval (a, b) satisfies the condition

$$(4) \qquad (b - a) = 2i\pi/k\,,$$

where i is any integer. It would seem that the error associated with the use of Eq. (2) as approximations for S and C would be intolerable. However, this suspicion is unfounded when $k \gg 1$. Let E_s and E_c be the error terms, thus

$$(5) \qquad S = S^* + E_s \quad \text{and} \quad C = C^* + E_c\,,$$

then it is shown below that

$$(6) \qquad |E_s| \leqq M(b - a)/k^3 \quad \text{and} \quad |E_c| \leqq M(b - a)/k^3$$

where

$$(7) \qquad M = \left| \max_{a \leqq x \leqq b} \left\{ \frac{d^3 f}{dx^3} \right\} \right|.$$

Since the error is proportional to k^{-3}, it will be small for large k, provided that the third derivative of $f(x)$ is not large relative to k.

Thus, there is an interesting set of circumstances under which accurate estimates of the integrals in Eq. (1) can be obtained by a trivial computation. This work was stimulated by some recent work of Clendenin [2], who indicated that formulas of the type shown in Eq. (2) would not be very well suited for practical computations. Since this was not supported by a computation of the error bounds, we decided to

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determine them. Since analysis of the error in Filon quadrature is rare, Luke's work [3] being one of the rare cases, many details are given here.

The Filon Formula. This quadrature formula is derived as follows. In Eq. (1) $f(x)$ is replaced by a polynomial approximation, in particular a second-degree polynomial which agrees with $f(x)$ at three points. Since the integrals

$$S = \int_a^b x^m \sin(kx)dx\,, \qquad C = \int_a^b x^m \cos(kx)dx\,, \tag{8}$$

are obtainable in closed form this procedure leads to a quadrature formula. Following the usual pattern in constructing quadrature formulas, the interval of integration (a, b) is subdivided into p panels, each of length $2h$. The integration formula is applied to each panel; in this application the polynomial approximation of $f(x)$ is required to agree with $f(x)$ at the endpoints and midpoint of the panel. Finally the sum of the contributions from each panel gives the desired quadrature formula. These formulas are

$$S = h[\alpha(f_0 \cos kx_0 - f_{2p} \cos kx_{2p}) + \beta S_{2p} + \gamma S_{2p-1}] + E_s\,, \tag{9}$$

$$C = h[\alpha(f_{2p} \sin kx_{2p} - f_0 \sin kx_0) + \beta C_{2p} + \gamma C_{2p-1}] + E_c\,, \tag{10}$$

where

$$S_{2p} = \sum_{i=0}^{p} f_{2i} \sin kx_{2i} - \tfrac{1}{2}[f_0 \sin kx_0 + f_{2p} \sin kx_{2p}]\,, \tag{11}$$

$$S_{2p-1} = \sum_{i=1}^{p} f_{2i-1} \sin kx_{2i-1}\,, \tag{12}$$

$$C_{2p} = \sum_{i=0}^{p} f_{2i} \cos kx_{2i} - \tfrac{1}{2}[f_0 \cos kx_0 + f_{2p} \cos kx_{2p}]\,, \tag{13}$$

$$C_{2p-1} = \sum_{i=1}^{p} f_{2i-1} \cos kx_{2i-1}\,, \tag{14}$$

$$\alpha = 1/\theta + (\sin 2\theta)/2\theta^2 - (2 \sin^2 \theta)/\theta^3\,, \tag{15}$$

$$\beta = 2((1 + \cos^2 \theta)/\theta^2 - (\sin 2\theta)/\theta^3)\,, \tag{16}$$

$$\gamma = 4((\sin \theta)/\theta^3 - (\cos \theta)/\theta^2)\,, \tag{17}$$

$$\theta = kh\,, \tag{18}$$

$$f_i = f(x_i)\,, \qquad x_{i+1} - x_i = h\,, \qquad x_0 = a\,, \qquad x_{2p} = b\,, \tag{19}$$

and E_s and E_c are the error terms associated with using the first term on the right of Eqs. (9) and (10) as an approximation for S and C. Where θ is small it is necessary to replace the expressions for α, β, and γ by power series in θ to avoid the loss of significant figures due to cancellation in these expressions; this fact has been overlooked in a recently published algorithm [4].

The Error Term. Peano's theorem [5] is used to put the error terms E_s and E_c in a useful form. Define the operator

$$L(f) = \int_a^b f(x)t(x)dx - \sum_{i=1}^{j} b_i f(x_i) , \tag{20}$$

where $L(f)$ vanishes when f is a polynomial of degree n or less, the x_i are contained in the closed interval $[a, b]$, $t(x)$ is piecewise continuous in this interval, and the $(n + 1)$th derivative of $f(x)$ is continuous in this interval. Then, by Peano's theorem,

$$L(f) = \int_a^b f^{(n+1)}(t)K(t)dt , \tag{21}$$

where $K(t)$, the Peano kernel, is given by

$$K(t) = (1/n!)L_x((x - t)_+^n) , \tag{22}$$

and

$$(x - t)_+^n = (x - t)^n , \quad x \geqq t , \qquad (x - t)_+^n = 0 , \quad x < t , \tag{23}$$

and the subscript x in L_x denotes that x, rather than t, is regarded as the variable. In the present application E_s is to be identified with $L(f)$ to obtain a useful expression for the error in the quadrature formula for the sine integral; similarly, E_c is identified with $L(f)$ for the error in the cosine integral.

Let us now direct our attention to applying the quadrature formula on one panel; thus we use Eqs. (9) and (10) with $p = 1$. The contribution to S from one panel is

$$S_i = \int_{x_i}^{x_i+2h} f(x) \sin (kx)dx , \tag{24}$$

and the contribution to C from one panel is

$$C_i = \int_{x_i}^{x_i+2h} f(x) \cos (kx)dx . \tag{25}$$

After a change of variables S_i and C_i can be expressed as follows:

$$\begin{aligned} S_i &= \cos (kx_i + \theta) \int_{-h}^{+h} g_i(z) \sin (kz)dz + \sin (kx_i + \theta) \int_{-h}^{+h} g_i(z) \cos (kz)dz , \\ C_i &= \cos (kx_i + \theta) \int_{-h}^{+h} g_i(z) \cos (kz)dz - \sin (kx_i + \theta) \int_{-h}^{+h} g_i(z) \sin (kz)dz , \end{aligned} \tag{26}$$

where

$$g_i(z) = f(z + x_i + h) . \tag{27}$$

Let us define the two integrals appearing in these expressions as

$$s_i = \int_{-h}^{+h} g_i(z) \sin (kz)dz , \qquad c_i = \int_{-h}^{+h} g_i(z) \cos (kz)dz , \tag{28}$$

and consider applying the Filon quadrature formula to them. The task now is to determine the Peano kernel for each case; first we consider s_i. Identifying the operator L with the error term associated with using the Filon quadrature formula for s_i we have

$$L(g_i) = \int_{-h}^{+h} g_i(z) \sin (kz)dz + h\left(\alpha \cos \theta - \frac{\beta}{2} \sin \theta\right)(g_i(h) - g_i(-h)) . \tag{29}$$

It is clear that $L(g_i)$ is zero when $g_i(z)$ is a polynomial of second degree, since the quadrature formula is designed to be exact in this case. On the other hand it is not exact for a polynomial of third degree, as may be verified by substituting z^3 for $g_i(z)$ in Eq. (29); this is different from the situation for Simpson's rule which is also designed to be exact for polynomials of second degree, but which is exact for polynomials of third degree too. Consequently, the Peano kernel is

$$K_s(t) = \tfrac{1}{2}\left\{\int_{-h}^{+h} (x-t)_+^2 \sin(kx)dx + h\left(\alpha\cos\theta - \frac{\beta}{2}\sin\theta\right) \times ((h-t)_+^2 - (-h-t)_+^2)\right\}. \tag{30}$$

Notice that

$$(-h-t)_+ = 0\,, \quad -h \leqq t \leqq h\,, \quad (h-t)_+ = h - t\,, \quad -h \leqq t \leqq h\,, \tag{31}$$

hence the kernel can be written

$$K_s(t) = \tfrac{1}{2}\left\{\int_t^h (x-t)^2 \sin(kx)dx + h\left(\alpha\cos\theta - \frac{\beta}{2}\sin\theta\right)(h-t)^2\right\}. \tag{32}$$

After executing the integration and some algebraic manipulations this becomes

$$K_s(t) = h^3\left\{\frac{\sin\theta}{2\theta^2}\left(1 - \left(\frac{t}{h}\right)^2\right) + \frac{1}{\theta^3}\left(\cos\theta - \cos\left(\frac{\theta t}{h}\right)\right)\right\}. \tag{33}$$

Similarly, one obtains for the kernel associated with the next integral in Eq. (28)

$$K_c(t) = \frac{h^4}{6}\left\{2\left(\frac{\sin\theta}{\theta^3} - \frac{\cos\theta}{\theta^2}\right)\left(\left(1-\frac{t}{h}\right)^3 - 2\left(-\frac{t}{h}\right)_+^3\right) + 3\frac{\cos\theta}{\theta^2}\left(1-\frac{t}{h}\right)^2 - \frac{6\sin\theta}{\theta^3}\left(1-\frac{t}{h}\right) - \frac{6}{\theta^4}\left(\cos\theta - \cos\left(\frac{\theta t}{h}\right)\right)\right\}. \tag{34}$$

It will be noticed that $K_c(t)$ contains a multiplier h^4 instead of h^3; this arises from the fact that the Filon formula is exact for c_i, Eq. (28), when $g_i(z) = z^3$. Thus in this case the situation is analogous to Simpson's rule. Equation (34) differs from Luke's result (Eq. (19) in [3]) which contains an error. A term $-2(-t/h)_+^3$ in Eq. (34) was omitted by Luke. This omission stems from the omission of the $j = 0$ term in Eq. (12) of his paper. As a result the regions of definiteness for $G_1(s, \theta)$ and $G_2(s, \theta)$ in his paper are incorrect, but the error curves shown in Figs. 1–5 in his paper are correct.*

The Simple Formulae. It is apparent that under the conditions cited in the introduction, Eqs. (3) and (4), the simple expressions in Eq. (2) result from Eqs. (9) and (10); notice that Eq. (4) implies that θ is an integral multiple of π,

$$\theta = n\pi\,. \tag{35}$$

Recalling the definition of S_i, Eq. (26), it is seen that

$$S_i = \cos((m + (i+1)n)\pi) \int_{-h}^{+h} g_i(z)\sin(kz)dz\,, \tag{36}$$

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hence only the kernel $K_s(t)$ enters into a consideration of the error associated with applying the Filon formula to this panel. Since we now have $\theta = n\pi$, substitution of this value in $K_s(t)$, Eq. (33), yields

$$K_s(t) = (h^3/n^3\pi^3)\,(\cos n\pi - \cos\,(n\pi t/h))\,. \tag{37}$$

The quadrature error is given by Eq. (21), hence for this case

$$\begin{aligned} L(g_i) = \cos\,((m + (i+1)n)\pi) \int_{-h}^{+h} g_i^{(3)}(t) \\ \times \left\{\left(\frac{h}{n\pi}\right)^3 \left(\cos n\pi - \cos\left(\frac{n\pi t}{h}\right)\right)\right\} dt\,. \end{aligned} \tag{38}$$

Since the kernel does not change sign over the interval of integration, the mean-value theorem can be applied to obtain

$$\begin{aligned} L(g_i) = g_i^{(3)}(\xi) \cos\,((m + (i+1)n)\pi) \int_{-h}^{+h} \left(\frac{h}{n\pi}\right)^3 \\ \times \left(\cos n\pi - \cos\left(\frac{n\pi t}{h}\right)\right) dt\,, \qquad -h < \xi < h\,, \end{aligned} \tag{39}$$

and, performing the integration,

$$L(g_i) = g_i^{(3)}(\xi) \cos\,((m + (i+1)n)\pi)\,(h/n\pi)^3(2h)\cos\,(n\pi)\,. \tag{40}$$

Summation of $L(g_i)$ over the p panels yields the error E_s. Using the definition of θ, Eq. (35), and the fact that $2hp = b - a$, the inequality (6) results.

A parallel calculation yields the same inequality for the error in the cosine term. The kernel $K_c(t)$ does not enter in this computation, since the coefficient of the cosine integral in Eq. (26) vanishes.

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