An Error Analysis for Numerical Multiple Integration. I

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1. Introduction. Error analyses for numerical methods dealing with functions of more than one variable are not abundant in the literature. The purpose of this paper is to popularize and to extend an idea due to Davis [12] for estimating the error made in approximating analytic functions of more than one variable. Two types of asymptotic results are given for the new cubatures, as well as numerical examples.

Sard [24] has obtained error estimates that involve various partial derivatives of the function f to be integrated. His kernel theorem for functions of two variables is established by noting the effect of an error functional on the remainder in a Taylor's expansion of f. He obtains sharp bounds for the appropriate function spaces, but these bounds are frequently inconvenient to apply because of the difficulty in computing them. For the one-dimensional case, see Stroud and Secrest [29, p. 65]. In one dimension, Davis [12] has stated a method of estimation for analytic functions that has the advantage of being comparatively easy to compute. An interesting feature of Davis' work is that it can be generalized to deal with analytic functions of more than one variable. He noted this in one paper [12] and it has been mentioned again by Valentin [30]. In this paper, Davis' method is extended and, in a future paper, it will be compared with Sard's method.

Other authors have studied error bounds for some special cases, which we now discuss. Error estimates for cross-product rules have been given by several authors [20], [25], [29] and the general idea has been to express the error of the cross-product rule as the product of the errors of lower-dimensional rules. Variations of this procedure include, for example, Hammer's conical product rules [20]. Stenger [27] has recently considered error estimates for the cross-products of Gaussian quadratures. Von Mises has established a certain error bound for cubatures, which is discussed by Stroud [28] and involves bounding certain partial derivatives after a transformation into spherical coordinates. Lyness [22] has discussed symmetric integration rules and his work contains error estimates. During the work leading to this paper, the author conjectured that theorems similar to those proved in Krylov [21] for the trapezoidal and Simpson's rules and in Meinguet [23] for Romberg one-dimensional integration could be proved for symmetric one-dimensional rules and extended to symmetric multidimensional rules by using Lyness' work. This conjecture has not been resolved. There is also a growing literature on approximation of functions of more than one variable by spline functions, references to which can be found in an article by Birkhoff and de Boor [9].

Although only cubatures will be discussed in this paper, the same methods can be used for other linear approximations, some of which will be discussed in a future

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paper. We assume that the functions to be integrated are analytic and uniformly bounded in norm. The goal is to find bounds on the error functional that are simpler than the standard ones in the sense of not involving various partial derivatives. In principle, this seems feasible because of Cauchy's integral formula for analytic functions, which relates derivatives to values of the function itself.

We shall consider cubatures of the form

$$\iint_{I\times I} f(x, u)dxdu = \sum_{k=1}^{n} A_k f(x_k, u_k)\,, \tag{1}$$

where $I \times I = \{(x, u)\colon -1 \leqq x \leqq 1,\ -1 \leqq u \leqq 1\}$. E_ρ is the ellipse with foci at ± 1, semimajor axis a, semiminor axis $b = (a^2 - 1)^{1/2}$, and $\rho = (a + b)^2$. $E_\rho \times E_\rho$ is called an elliptic bicylinder [10] and it consists of pairs of complex numbers (z, w) with z and w each belonging to the region enclosed by E_ρ. The more general elliptic bicylinder $E_\rho \times E_{\rho'}$ consists of (z, w) such that z is in the region enclosed by E_ρ and w is in the region enclosed by $E_{\rho'}$. We consistently use the notation $z = x + iy$, $w = u + iv$, where x, y, u and v are real. The functions to be integrated, $f(x, u)$, are in the space $L^2(E_\rho \times E_\rho) = \{f(z, w)\colon f$ is analytic for (z, w) inside $E_\rho \times E_\rho$ and $\int\int\int\int_{E_\rho\times E_\rho} |f(z, w)|^2 dxdydudv$ exists$\}$. Let us denote $L^2(E_\rho \times E_\rho)$ by L^2.

2. Properties of L^2. The Hilbert space $L^2(E_\rho) = \{f(z)\colon f$ is analytic inside E_ρ and $\int\int_{E_\rho} |f(z)|^2 dxdy$ exists$\}$ is discussed in Davis [14]. It has been used in quadrature theory by various authors [1]–[7], [12], [13], [16], [30], [31] and it is to be noted that analogues of certain properties of $L^2(E_\rho)$ that are applicable to quadratures also hold for L^2 and cubatures. Some of these properties are summarized in Theorem 1.

THEOREM 1. *L^2 is a Hilbert space in which point functionals are bounded. If $\{p_r(z)\}_{r=0}^{\infty}$ is a complete orthonormal sequence in $L^2(E_\rho)$, then $\{p_r(z)p_s(w)\}_{r,s=0}^{\infty}$ is a complete orthonormal sequence in L^2.*

Proof. L^2 is an inner product space with the inner product

$$(f, g) = \iiiint_{E_\rho \times E_\rho} f(z, w)\overline{g(z, w)}\, dxdydudv$$

and the norm $\|f\| = (f, f)^{1/2}$ [8]. In order to show that L^2 is complete, we first show that point functionals are bounded. For the point functional $L(f) = f(0, 0)$, the inequality

$$|f(0, 0)|^2 \leqq \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} |f(\rho_1 e^{i\alpha_1}, \rho_2 e^{i\alpha_2})|^2 d\alpha_1 d\alpha_2$$

follows from Cauchy's integral formula, where $z = \rho_1 e^{i\alpha_1}$, $w = \rho_2 e^{i\alpha_2}$ and the circular bicylinder $A\colon 0 \leqq \rho_j \leqq r$, $j = 1, 2$, is chosen so that $f(z, w)$ is analytic in it. Multiplication of both sides of the inequality by $\rho_1\rho_2\, d\rho_1 d\rho_2$ and integration bebetween $0 \leqq \rho_j \leqq r$, $j = 1, 2$, yields the following:

$$|f(0, 0)|^2 \leqq \frac{1}{\pi^2 r^4} \int_A |f(z, w)|^2 dv_x dv_y$$

where $dv_x dv_y = dxdudydv$. Thus

$$|f(0,0)| \leqq \frac{1}{\pi r^2}\left[\int_A |f(z,w)|^2 dv_x dv_y\right]^{1/2} \leqq \frac{1}{r^2}\|f\|$$

and the point functional $L(f) = f(0, 0)$ is bounded. In order to prove that the point functional $L(f) = f(z_0, w_0)$ is bounded for an arbitrary (z_0, w_0) in $E_\rho \times E_\rho$, it suffices to choose $r > 0$ such that the circular bicylinder $|z - z_0| \leqq r$, $|w - w_0| \leqq r$ is contained in $E_\rho \times E_\rho$; then the above proof can be paraphrased to get the desired result. Following Bochner and Martin [10], we write $|f(z, w)| \leqq w_r\|f\|$ where w_r depends on r but not on f. The fact that point functionals are bounded means that norm convergence implies uniform convergence on closed subsets of $E_\rho \times E_\rho$. For, if $\{f_n\}$ is a Cauchy sequence in L^2 and B is an arbitrary closed subset of $E_\rho \times E_\rho$ with (z, w) in B, then the inequality

$$|f_n(z,w) - f_m(z,w)| \leqq w_r\|f_n - f_m\|$$

implies that $\{f_n\}$ is a Cauchy sequence in the sup norm on B. Since the f_n are analytic, they have a limit function f which is also analytic and, since B is an arbitrary closed subset of $E_\rho \times E_\rho$, f is analytic in $E_\rho \times E_\rho$. The fact that f has finite norm can be proved directly by considering a sequence of closed sets that fill up $E_\rho \times E_\rho$ [10]. A second proof can be made by considering the space of functions of finite norm but which need not be analytic, and by using the theorem that L^2 convergence implies almost everywhere pointwise convergence [10].

The proof that L^2 is a Hilbert space is similar to the proof that $L^2(E_\rho)$ is a Hilbert space and is omitted. If $\{p_r(z)\}_{r=0}^\infty$ is a complete orthonormal sequence in $L^2(E_\rho)$, then it follows that $\{p_r(z)p_s(w)\}_{r,s=0}^\infty$ is an orthonormal sequence in L^2. The proof of the completeness of this sequence is analogous to that given in Courant-Hilbert [11] for a real inner product space. Q.E.D.

It should be noted that the analogous theorem holds for the more general space $L^2(E_\rho \times \cdots \times E_\rho) = \{f(z) : f$ is analytic in the k complex variables $z = (z_1, \cdots, z_k)$ and $\int\int_{E_\rho\times\cdots\times E_\rho} |f(z)|^2 dv_x dv_y$ exists$\}$, where $z = x + iy$, $x = (x_1, \cdots, x_k)$, $y = (y_1, \cdots, y_k)$, $dv_x = dx_1 \cdots dx_k$, and $dv_y = dy_1 \cdots dy_k$. A generalized Cauchy formula yields an inequality of the form $|f(z)| \leqq w\,\|f\|$, where z is the vector $(z_1, \cdots, z_k)$ and w depends on the distance from z to the boundary of $E_\rho \times \cdots \times E_\rho$ but not on f. Hence, point functionals are bounded, and the rest of the proof proceeds as before. The complete orthonormal sequence is $\{p_{r_1}(z_1)p_{r_2}(z_2)\cdots p_{r_k}(z_k)\}_{r_1,\cdots,r_k=0}^\infty$.

3. Bounded Linear Functionals Defined on L^2. In the cubature methods to be described, we need the representers of certain bounded linear functionals and the norms of these functionals. Since point functionals are bounded in L^2, we note that L^2 has a reproducing kernel function of the form

$$K(z, w; s, t) = \sum_{k=0}^{\infty} h_k(z,w)\overline{h_k(s,t)}\,, \tag{2}$$

where z, w, s and t are complex variables and $\{h_k\}$ is a complete orthonormal sequence in L^2.

THEOREM 2. *Let $\{h_k\}_{k=0}^\infty$ be a complete orthonormal sequence for L^2, and suppose that L is a bounded linear functional on L^2. Then the representer of L and its norm are given by the following equations:*

(3) $$h(z, w) = \sum_{k=0}^{\infty} h_k(z, w)\overline{L(h_k)},$$

(4) $$\|L\|^2 = \sum_{k=0}^{\infty} |L(h_k)|^2.$$

Proof. Equation (4) holds for a bounded linear functional in any Hilbert space in which $\{h_k\}$ is a complete orthonormal sequence. The proof of Eq. (3) is analogous to that given by Davis [14] for functions of one complex variable. By using the fact that norm convergence implies uniform convergence on closed subsets of $E_\rho \times E_\rho$ and by making a useful choice of Fourier coefficients, Bochner and Martin show that $\sum_{k=0}^{\infty} |h_k(z, w)|^2 \leqq w_r^2$ for (z, w) a distance of at least r from the boundary of $E_\rho \times E_\rho$, where w_r depends only on r. From this inequality it can be shown that the function K defined by Eq. (2) is analytic in each pair of variables and that it has the reproducing property

$$f(s, t) = (f(z, w), K(z, w; s, t))$$

where the inner product is taken with respect to (z, w). The representer of L is determined as follows: $L[f(s, t)] = (f(z, w), L[K(z, w; s, t)]) = (f(z, w), h(z, w))$ where $h(z, w) = L_{(s,t)}[K(z, w; s, t)]$ is the representer of L. The subscript (s, t) means that L is applied to $K(z, w; s, t)$ considered as a function of (s, t) only. Since the reproducing kernel converges uniformly and absolutely in closed subsets of $E_\rho \times E_\rho$,

$$L_{(s,t)}\left[\sum_k h_k(z, w)\overline{h_k(s, t)}\right] = \sum_k h_k(z, w)\overline{L_{(s,t)}[h_k(s, t)]}$$

which is Eq. (3). Q.E.D.

If we want to consider functions of n complex variables, then we write the reproducing kernel as

(2′) $$K(z; s) = \sum_k h_k(z)\overline{h_k(s)}$$

where z and s are n-dimensional complex variables. The representer of L is given by

(3′) $$h(z) = \sum_k h_k(z)\overline{L(h_k)}$$

and Eq. (4) holds as it is.

4. Application to Cubatures. Theorem 2 is used to obtain a bound on the cubature error as follows: Let

(5) $$R_n(f) = \iint_{I\times I} f(x, u)\,dx\,du - \sum A_k f(x_k, u_k)$$

for f in L^2. Then $|R_n(f)| \leqq \|R_n\| \cdot \|f\|$, by the definition of the norm of R_n, and since R_n is bounded in L^2, $\|R_n\|$ is finite. Let us assume that we have some particular function f in L^2 and a bound on its norm, $\|f\| \leqq r$ for some r, and let S_r denote the set of g in L^2 such that $\|g\| \leqq r$. Then the inequality

$$|R_n(g)| \leqq ||R_n|| \cdot r \tag{6}$$

is sharp on S_r; that is, it holds for all g in S_r and is achieved by at least one of them. Davis' idea, for a similar Hilbert space, was to use the inequality (6), with $||R_n||$ computed by Eq. (4), where R_n is determined by the cubature rule used.

It should be noted that the functional $L_T(f) = \int\int_T f$ is bounded on L^2 if T is a closed subregion of $E_\rho \times E_\rho$. (This can be proved by arguments similar to those in Davis [14] for the one-dimensional case.) In particular, if T is a closed real subset of $E_\rho \times E_\rho$, then L_T is bounded, so that the cubatures of interest need not, in principle, be restricted to rectangular regions. The limitation on T, in practice, is that it must be possible to calculate $L_T(p_r p_s)$, where $p_r p_s$ is a member of the complete orthonormal sequence in L^2.

In order to obtain the needed formulas for R_n as defined in (5), we note that

$$||R_n||^2 = \sum_{r,s=0}^{\infty} |R_n(p_r(z)p_s(w))|^2 \tag{7}$$

where $p_r(z) = 2\{(r+1)/[\pi(\rho^{r+1} - \rho^{-r-1})]\}^{1/2} U_r(z)$, $r = 0, 1, \cdots$, and $U_r(z)$ is the rth Tchebycheff polynomial of the second kind. This formula can be rewritten as

$$||R_n||^2 = \sum_{r,s} \alpha(r,\rho)\alpha(s,\rho) \left| \beta(r)\beta(s) - \sum_{k=1}^{n} A_k U_r(x_k) U_s(u_k) \right|^2 \tag{7'}$$

where $\alpha(r, \rho) = 4(r+1)/[\pi(\rho^{r+1} - \rho^{-r-1})]$ and $\beta(r) = [1 + (-1)^r]/(r+1)$, $r = 0, 1, \cdots$. Thus $||R_n||$ can be computed once and for all for any given set of cubature weights A_k and base points (x_k, u_k).

The generalization of this method to higher dimensions is not difficult. Let I_l denote the hypercube $|x_k| \leqq 1$, $k = 1, \cdots, l$. For l-fold integral $\int_{I_l} f(x)dx$, $x = (x_1, \cdots, x_l)$, we use a cubature of the form $\sum_{k=1}^{n} A_k f(x^k)$, where x is l-dimensional. The expression for $||R_n||^2$ becomes

$$\sum_{r_1, r_2, \cdots, r_l = 0}^{\infty} |R_n[p_{r_1}(z_1) p_{r_2}(z_2) \cdots p_{r_l}(z_l)]|^2$$

where the $p_{r_j}(z)$ are the appropriately scaled Tchebycheff polynomials of the second kind.

The bound on $|R_n(f)|$ given by Eqs. (6) and (7) have been calculated for integration over the two-dimensional square $-1 \leqq x,\ u \leqq 1$ for certain known cubature rules. These numerical results are given in Section 9.

We noted that formula (7) holds for numerical integration over more general regions than the square, if R_n is redefined appropriately. That is, if the desired integral is $\int\int_T f$ over the real two-dimensional closed and bounded region T, then formula (7′) is correct if $\int\int_T U_r(x) U_s(u) dx\, du$ is substituted for $\beta(r)\beta(s)$ in (7′).

5. Cubatures with Remainders of Minimum Norm. The idea is to minimize $||R_n||$ in Eq. (7′) by an appropriate choice of the weights A_k and, sometimes, the base points (x_k, u_k). We remark that, so long as the A_k and (x_k, u_k) are considered as being in some compact set in complex $3n$-dimensional space, $||R_n||$ has a (local) minimum. Minimization with respect to the A_k yields the following system of equations:

$$\sum_{r,s} \alpha(r, \rho)\alpha(s, \rho)\left[\beta(r)\beta(s) - \sum_l A_l U_r(x_l) U_s(u_l)\right] U_r(x_k) U_s(u_k) = 0\,, \tag{8}$$
$$k = 1, \cdots, n\,.$$

Minimization with respect to the x_k and u_k yields the following systems:

$$\begin{aligned} &\sum_{r,s} \alpha(r, \rho)\alpha(s, \rho)\left[\beta(r)\beta(s) - \sum_l A_l U_r(x_l) U_s(u_l)\right] A_k U_r'(x_k) = 0\,, \\ &\sum_{r,s} \alpha(r, \rho)\alpha(s, \rho)\left[\beta(r)\beta(s) - \sum_l A_l U_r(x_l) U_s(u_l)\right] A_k U_s'(u_k) = 0\,, \\ &\qquad k = 1, \cdots, n\,. \end{aligned} \tag{9}$$

If the points are assumed given, then only system (8) is to be solved and, moreover, it is a linear system in the A_k, so that it might appear that there would be comparatively little computational difficulty. The numerical results in this paper are only for this linear case, and it has turned out to be difficult to sum the double series involved accurately. This point, as well as a solution of the nonlinear case (i.e., systems (8) and (9)), will be considered by the author and Steven L. Shrier in a forthcoming paper which will deal with the computational aspects of the minimum norm (MN) cubatures. We only remark here that the main computational difficulties that arise in the solution of (8) are the difficulties in summing the infinite series and the relative instability of the coefficient matrix for large ρ. These two factors work against one another, since the series are easier to sum (i.e., require fewer terms) for larger ρ.

6. Asymptotic Properties of the MN Cubatures. Let R_n^{MN} denote the error functional corresponding to the MN cubature (in two variables) with n points, and let R_n^G denote the error functional corresponding to the cross-product rule G formed from any two Gaussian rules, G_1 and G_2, such that the number of points in the cross-product rules is greater than or equal to n. The next theorem gives a bound on $||R_n^G||$ in terms of n.

THEOREM 3. *For the space L^2,*

$$||R_n^G||^2 \leqq \sum_{r,s=0;\, r,s \text{ even}}^{\infty} \left\{ \frac{2}{r+1} \gamma(N_2)[\alpha(s, \rho)\psi(s)]^{1/2} + 2\gamma(N_1)[\alpha(r, \rho)\psi(r)]^{1/2}(s+1) \right\}^2,$$

where

$$n = N_1 \cdot N_2\,, \qquad \alpha(s, \rho) = 4(s+1)/[\pi(\rho^{s+1} - \rho^{-s-1})]\,,$$
$$\psi(s) = \{(s+1)[(s+1)^2 - 1] \cdots [(s+1)^2 - (2N_2)^2]\}^2\,,$$

and

$$\gamma(N) = \frac{2^{2N+1}}{(2N+1)(2N)!}\left\{\frac{(N!)^2}{(2N)!}\right\}^2 \frac{1}{1\cdot 3 \cdots (4N+1)}\,.$$

Proof. By Eq. (7) $||R_n^G||^2 = \sum |R_n^G(p_r(z)p_s(w))|^2$. $R_n^G(p_r, p_s) = \iint p_r p_s - Q^G(p_r p_s)$, where $Q^G(f) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} A_i B_j f(x_i, u_j)$, N_i being the number of points corresponding to rule G_i, $i = 1, 2$, and A_i, B_j, x_i, u_j being the appropriate weights and base points for the (one-dimensional) Gaussian rules G_1 and G_2, respectively.

$$\begin{aligned} R_n^G(p_r p_s) &= \int p_r \int p_s - \sum_i A_i p_r(x_i) \sum_j B_j p_s(u_j) \\ &= \int p_r \left\{ \int p_s - \sum_j B_j p_s(u_j) \right\} + \int p_r \sum_j B_j p_s(u_j) \\ &\quad - \sum_i A_i p_r(x_i) \sum_j B_j p_s(u_j) \\ &= \beta_r R^2(p_s) + R^1(p_r) \sum_j B_j p_s(u_j) , \end{aligned}$$

where R^i is the error functional corresponding to G_i, $i = 1, 2$, and $\beta_r = [1 + (-1)^r]/(r + 1)$, $r = 0, 1, \cdots$. (We observe that an analogous expression can be derived by adding and subtracting the corresponding terms in the other variable above.) Now

$$|R_n^G(p_r p_s)| \leqq \beta_r |R^2(p_s)| + 2|R^1(p_r)| \max_j |p_s(u_j)| \tag{10}$$

where we have used the fact that the Gauss weights are positive and add up to the length of the interval. We now note the fact that the Gauss points are in $[-1, 1]$ and that the bound $|U_r(x)| \leqq r + 1$ holds for x in $[-1, 1]$. The author [7] has previously obtained a bound on $|R(p_r)|$, $r = 0, 1, \cdots$, where R corresponds to an N-point Gauss rule, and a substitution of it into inequality (10) yields the result stated in the conclusion. Q.E.D.

In order to make the bound on $||R_n^G||$ given by Theorem 3 more meaningful, we remark that, by using fairly crude estimates, $\gamma(N)$ can be shown to be bounded by $2^{2N} N^{-4N-5}$.

We also remark that the proof of Theorem 3 is somewhat similar in spirit to results given by Stenger [27].

Since $||R_n^{MN}|| \leqq ||R_n^G||$, we note that Theorem 3 also yields a bound on $||R_n^{MN}||$. It is an open question as to how conservative this bound and the one of Theorem 3 are.

7. *MN* Rules for Other Rectangles. As mentioned earlier, formulas can be generated over other regions than the basic square $-1 \leqq x, u \leqq 1$. For use in composite rules, we mention an alternative approach for generating MN formulas for other rectangles than those tabulated for the basic square. If we consider

$$\int_a^b \int_c^d f(s, t) ds dt \simeq \sum_{k=1}^n B_k f(s_k, t_k) ,$$

then it can be shown that $B_k = [(b - a)/2][(d - c)/2]A_k$, $s_k = [(b - a)/2]x_k + (a + b)/2$ and $t_k = [(d - c)/2]u_k + (c + d)/2$, where the A_k, x_k and u_k refer to MN rules with the basic square. Also, if we denote by $R_n(a, b; c, d)$ the error in approximating $\int_a^b \int_c^d f$, then

$$|R_n(a, b; c, d)| \leqq [(b - a)/2][(d - c)/2] ||R_n|| \, ||g|| ,$$

where $||R_n||$ refers to the norm of the functional on the basic square and the function $g(z, w)$ is defined by

$$g(z, w) = f\left(\frac{b-a}{2}z + \frac{a+b}{2}, \frac{d-c}{2}w + \frac{c+d}{2}\right)$$

for (z, w) in $E_\rho \times E_\rho$.

8. Convergence of the Minimum Norm Weights. For the one-dimensional case, the following theorems are known [7]: For a fixed n, if the quadrature nodes are fixed, then the A_k^{MN} converge as $\rho \to \infty$ to the weights of the corresponding interpolatory quadrature. If the nodes are variable, then the weights and points of the minimum norm quadrature converge to the weights and points of the corresponding Gaussian quadrature. The theorem that follows generalizes the first of these two theorems. It is stated for two dimensions, but can be generalized to m dimensions, $m \geqq 2$.

THEOREM 4. *Given N points in the square $-1 \leqq x, u \leqq 1$, such that N is of the form*

$$\binom{d+2}{2}$$

and there exist $A_1, \cdots, A_N$ such that the corresponding cubature is of precision d, then the minimum norm cubature weights A_i^{MN} have the property that

$$A_i^{MN} \to A_i \quad as \quad \rho \to \infty, \qquad i = 1, \cdots, N.$$

Proof. The following result will be implied by a lemma: $\lim |R_N(U_r U_s)| = 0$ as $\rho \to \infty$, $0 \leqq r + s \leqq d$. We recall that U_r is a polynomial of degree r, and so the above is equivalent to the following system of equations:

$$\begin{aligned}
\sum \quad A_i^{MN} &= \alpha_{00} + \epsilon_{00}(\rho) \equiv m_{00} \\
\sum A_i^{MN} x_i &= \alpha_{10} + \epsilon_{10}(\rho) \equiv m_{10} \\
\sum A_i^{MN} u_i &= \alpha_{01} + \epsilon_{01}(\rho) \equiv m_{01} \\
&\vdots \\
\sum A_i^{MN} u_i^d &= \alpha_{0d} + \epsilon_{0d}(\rho) \equiv m_{0d}
\end{aligned}$$

where α_{ij} is $\int\int_{I\times I} x^i u^j$, ϵ_{ij} is $R_N(x^i u^j)$ and so all the $\epsilon_{ij}(\rho) \to 0$ as $\rho \to \infty$. The coefficient matrix M of the A_i^{MN} is nonsingular by hypothesis and so $A^{MN} = M^{-1}m$, where A^{MN} is the vector of A_i^{MN} and m the constant vector of the above system. Hence, $\lim A^{MN} = M^{-1} \lim m = M^{-1}\alpha = A$, where α is the vector of α_{ij} and A is the vector of interpolatory A_i. Q.E.D.

Remark. A result similar to Theorem 4 holds for any region for which an interpolatory cubature can be defined.

LEMMA. *If the hypotheses of Theorem 4 hold, then* $\lim \rho^{d-(r+s)} \cdot |R_N(U_r U_s)|^2 = 0$ *as $\rho \to \infty$, for all nonnegative integers r and s such that $0 \leqq r + s \leqq d$.*

Proof. Let us denote by R_N^I the remainder of the interpolatory cubature based on the given N nodes, and recall that it has precision d. Then $||R_N^{MN}||^2 \leqq ||R_N^I||^2$ implies that

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left(\frac{4}{\pi}\right)^2 \frac{(r+1)(s+1)|R_N{}^{MN}(U_rU_s)|^2}{(\rho^{r+1}-\rho^{-r-1})(\rho^{s+1}-\rho^{-s-1})}$$
$$\leqq \sum_{s=0}^{\infty}\sum_{r=d-s+1}^{\infty}\left(\frac{4}{\pi}\right)^2 \frac{(r+1)(s+1)|R_N{}^{I}(U_rU_s)|^2}{(\rho^{r+1}-\rho^{-r-1})(\rho^{s+1}-\rho^{-s-1})}.$$

We multiply by $\rho^d - \rho^{-d}$ and delete all but the first term on the left-hand side. On the right-hand side, we see that the lowest-order term in ρ is essentially $1/\rho$, so that the limit of the right side is zero, as $\rho \to \infty$. Therefore, $\lim\, (\rho^d - \rho^{-d})|R_N{}^{MN}(1)|^2 = 0$.

We next take $r = 1$, $s = 0$, multiply by $\rho^{d-1} - \rho^{-d+1}$ and, proceeding as before, we obtain

$$(\rho^{d-1} - \rho^{-d+1})|R_N{}^{MN}(U_1U_0)|^2 \to 0 \quad \text{as} \quad \rho \to \infty\,.$$

The general result is thus seen to be

$$(\rho^{d-(r+s)} - \rho^{-d+(r+s)})|R_N{}^{MN}(U_rU_s)|^2 \to 0 \quad \text{as} \quad \rho \to \infty$$

and this holds for all r and s such that $0 \leqq r + s \leqq d$. Q.E.D.

Theorem 4 has an interesting application to the computationally important question of whether a minimum norm cubature has positive weights.

COROLLARY. *Assume the hypotheses of Theorem 4. Then, for large enough ρ, the minimum norm weights have the same sign as the interpolatory weights.*

This corollary says that, for example, the one-dimensional minimum norm quadrature with variable base points has positive weights, for large enough ρ, since the Gaussian weights are positive.

9. Tables and Examples. Two four-point rules on the square $-1 \leqq x, u \leqq 1$ are considered. These are the cross-product trapezoidal rule $T_2 \times T_2$ and the cross-product Gaussian rule $G_2 \times G_2$, to which Tables 1 and 2 refer, respectively. Both of these rules are fully symmetric rules; that is, symmetric points are included and symmetric points have the same cubature weight. For a four-point rule, this means that there is only one weight and, for $T_2 \times T_2$ and $G_2 \times G_2$, this weight is 1.0. (We remark that for a fully symmetric region and fully symmetric base points, the minimum norm weights are fully symmetric.) The corresponding minimum norm weight is denoted by A^{MN}. Two functions are considered, $f_1(x, u) = e^{x+u}$ and $f_2(x, u) = \cos x \cos u$. The definitions of the remaining symbols in Tables 1 and 2 are the following: a is the semimajor axis of the ellipse E_ρ; E_i is $|R_4{}^{MN}(f_i)|$; F_i is an upper bound on $||f_i||$; B_i is $||R_4{}^{MN}|| \cdot F_i$, $i = 1, 2$.

One nine-point rule is considered, namely, Lyness' rule [15, p. 141]. This rule is fully symmetric with generating points as shown in Table 3. The minimum norm weights are listed in order corresponding to the generating points and the symbols in Table 3 correspond to those in Tables 1 and 2.

For these examples, a bound on $||f||$ is found by means of the inequality $||f|| \leqq \sup\, |f(z, w)|\pi ab$ where the sup is over the region $E_\rho \times E_\rho$. However, for these two examples, the functions are also analytic on the boundary of $E_\rho \times E_\rho$ and so the sup can be changed to a max, which is over the boundary points only. (This use of the maximum modulus theorem is frequently a considerable compu-

TABLE 1

$T_2 \times T_2$

a	A^{MN}	E_1	E_2	F_1	F_2	$\|\|R_4\|\|$	$\|\|R_4^{MN}\|\|$	B_1	B_2
1.2	9.68896 −02	4.60 +00	2.72 +00	2.76 +01	3.77 +00	6.05912 +00	1.51693 +00	4.18 +01	5.72 +00
1.5	4.31306 −01	1.41 +00	2.33 +00	1.06 +02	1.51 +01	8.90344 −01	5.50164 −01	5.82 +01	8.31 +00
2.0	7.91015 −01	2.01 +00	1.91 +00	5.94 +02	9.24 +01	1.81787 −01	1.59230 −01	9.46 +01	1.47 +01
5.0	9.95019 −01	3.96 +00	1.67 +00	1.70 +06	3.46 +05	3.46876 −03	3.45903 −03	5.86 +03	1.20 +03

TABLE 2

$G_2 \times G_2$

a	A^{MN}	E_1	E_2	F_1	F_2	$\|\|R_4\|\|$	$\|\|R_4^{MN}\|\|$	B_1	B_2
1.2	8.98805 −01	5.92 −01	3.08 −01	2.76 +01	3.77 +00	5.86832 −01	5.60504 −01	1.55 +01	2.11 +00
1.5	9.92361 −01	7.81 −02	4.54 −02	1.06 +02	1.51 +01	7.22665 −02	7.20285 −02	7.62 +00	1.09 +00
2.0	9.99541 −01	3.87 −02	2.52 −02	5.94 +02	9.24 +01	8.50849 −03	8.50681 −03	5.05 +00	7.86 −01
5.0	1.00000 +00	3.62 −02	2.39 −02	1.70 +06	3.46 +05	2.43460 −05	2.43460 −05	4.13 +01	8.43 +00

TABLE 3

Lyness Base points: (0.632456, 0.0), (1.0, 1.0), (0.0, 0.0)

a	A^{MN}	E_1	E_2	$\|\|R_4\|\|$	$\|\|R_4^{MN}\|\|$	B_1	B_2
1.2	9.96935 −01 6.52136 −02 −4.71065 −01	5.62 −01	1.08 −02	5.87 −01	4.79 −01	1.32 +01	1.81 +00
1.5	1.11771 +00 1.04345 −01 −8.93696 −01	2.91 −02	1.93 −03	3.65 −02	3.58 −02	3.79 +00	5.41 −01
2.0	1.11498 +00 1.10022 −01 −9.00103 −01	5.45 −03	6.64 −03	2.12 −03	2.11 −03	1.26 +00	1.95 −01
5.0	1.11120 +00 1.11093 −01 −8.89180 −01	8.32 −03	6.66 −03	8.62 −07	8.62 −07	1.46 +00	2.99 −01

tational aid.) Additional methods of bounding $||f||$ can be based on the work of Davis [13].

As mentioned in Section 5, more substantial numerical results will appear in a future paper.

10. Conclusions. The error bounds of the Davis type, including the minimum norm cubatures, are probably more practical than the standard error bounds, because they do not involve bounding various partial derivatives of the integrand. However, these bounds are only applicable to functions that are analytic on the region of integration and that are bounded in norm over some cross-product of ellipses containing the region of integration. Sard's error estimates are applicable to a much wider class of functions. (This is, of course, a reason for conjecturing that the Davis-type estimates will be better for analytic functions.)

We state one question that has arisen from this work. In the one-dimensional case, the minimum norm quadratures have been shown to converge to the Gaussian quadratures as $\rho \to \infty$ (i.e., for a fixed n, the weights and points of the minimum norm rules converge to the corresponding weights and points of the Gaussian rule). However, the algebraic argument leading to this result breaks down in more than one dimension, although partial results have been obtained, as stated in Theorem 4, and so the question is: what is the asymptotic behavior of the cubature points?

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1. R. E. Barnhill, *Numerical Contour Integration* U. S. Army Mathematics Research Center Report No. 519, Madison, Wis., 1964.
2. R. E. Barnhill, "Complex quadratures with remainders of minimum norm," *Numer. Math.*, v. 7, 1965, pp. 384–390. MR **32** #8497.
3. R. E. Barnhill & J. A. Wixom, "Quadratures with remainders of minimum norm. I," *Math. Comp.*, v. 21, 1967, pp. 66–75.
4. R. E. Barnhill & J. A. Wixom, "Quadratures with remainders of minimum norm. II," *Math. Comp.*, v. 21, 1967, pp. 382–387.
5. R. E. Barnhill, "Optimal quadratures in $L^2(E_\rho)$. I," *SIAM J. Numer. Anal.*, v. 4, 1967, pp. 390–397.
6. R. E. Barnhill, "Optimal quadratures in $L^2(E_\rho)$. II." *SIAM J. Numer. Anal.* (To appear.)
7. R. E. Barnhill, "Asymptotic properties of minimum norm and optimal quadratures." (Submitted for publication.)
8. S. Bergman, *The Kernel Function and Conformal Mapping*, Math. Surveys, No. 5, Amer. Math. Soc., Providence, R. I., 1950. MR **12**, 402.
9. G. Birkhoff & C. R. deBoor, "Piecewise polynomial interpolation and approximation," *Approximation of Functions*, edited by H. L. Garabedian, Elsevier, Amsterdam, 1965. MR **32** #2789.
10. S. Bochner & W. T. Martin, *Several Complex Variables*, Princeton Math. Series, Vol. 10, Princeton Univ. Press, Princeton, N. J., 1948. MR **10**, 366.

11. R. Courant & D. Hilbert, *Methods of Mathematical Physics*, Vol. 1, Interscience, New York, 1953. MR **16**, 426.

12. P. J. Davis, "Errors of numerical approximation for analytic functions," *J. Rational Mech. Anal.*, v. 2, 1953, pp. 303–313. MR **14**, 907.

13. P. J. Davis, "Errors of numerical approximation for analytic functions," *Survey of Numerical Analysis*, McGraw-Hill, New York, 1962. MR **24** B1766.

14. P. J. Davis, *Interpolation and Approximation*, Blaisdell, New York, 1963. MR **28** #393.

15. P. J. Davis & P. Rabinowitz, *Numerical Integration*, Blaisdell, New York, 1967.

16. P. J. Davis & P. Rabinowitz, "On the estimation of quadrature errors for analytic functions," *MTAC*, v. 8, 1954, pp. 193–203. MR **16**, 404.

17. M. Golomb & H. F. Weinberger, "Optimal approximation and error bounds," *On Numerical Approximation*, edited by R. E. Langer, Proceedings of a Symposium, Univ. of Wisconsin Press, Madison, Wis., 1959. MR **22** #12697.

18. M. Golomb, *Lectures on Theory of Approximation*, Argonne National Laboratory, Argonne, Ill., 1962.

19. P. C. Hammer & A. H. Stroud, "Numerical evaluation of multiple integrals. II," *MTAC*, v. 12, 1958, pp. 272–280. MR **21** #970.

20. P. C. Hammer, "Numerical evaluation of multiple integrals," *On Numerical Approximation*, edited by R. E. Langer, Univ. of Wisconsin Press, Madison, Wis., 1959.

21. V. I. Krylov, *Approximate Calculation of Integrals*, translated from Russian, Macmillan, New York, 1962. MR **26** #2008.

22. J. N. Lyness, "Symmetric integration rules for hypercubes. I: Error coefficients," *Math. Comp.*, v. 19, 1965, pp. 260–276. MR **34** #952.

23. J. Meinguet, "Methods for estimating the remainder in linear rules of approximation. Application to the Romberg algorithm," *Numer. Math.*, v. 8, 1966, pp. 345–366. MR **33** #8102.

24. A. Sard, *Linear Approximation*, Math. Surveys, No. 9, Amer. Math. Soc., Providence, R. I., 1963. MR **28** #1429.

D. D. Stancu, "The remainder of certain linear approximation formulas in two variables," *SIAM J. Numer. Anal.*, v. 1, 1964, pp. 137–163. MR **31** #1503.

26. F. Stenger, "Bounds on the error of Gauss-type quadratures," *Numer. Math.*, v. 8, 1966, pp. 150–160. MR **33** #5120.

27. F. Stenger, "Error bounds for the evaluation of integrals by repeated Gauss-type formulae," *Numer. Math.*, v. 9, 1966, pp. 200–213.

28. A. H. Stroud, "Quadrature methods for functions of more than one variable," *Ann. New York Acad. Sci.*, v. 86, 1960, pp. 776–791. MR **22** #10179.

29. A. H. Stroud & D. Secrest, *Gaussian Quadrature Formulas*, Prentice-Hall, Englewood Cliffs, N. J., 1966.

30. R. A. Valentin, *Applications of Functional Analysis to Optimal Numerical Approximation for Analytic Functions*, Ph.D. Thesis, Brown University, Providence, R. I., 1965.

31. H. Yanagihara, "A new method of numerical integration of Gaussian type," *Bull. Fukuoka Gakugei Univ.* III, v. 6, 1956, pp. 17–24. (Japanese) MR **26** #5729.

32. G. Hämmerlin, "Über ableitungsfreie Schranken für Quadraturfehler," *Numer. Math.*, v. 5, 1963, pp. 226–233. MR **28** #1756.

33. G. Hämmerlin, "Über ableitungsfreie Schranken für Quadraturfehler. II. Ergänzungen und Möglichkeiten zur Verbesserung," *Numer. Math.*, v. 7, 1965, pp. 232–237. MR. **32** #1899.

34. G. Hämmerlin, "Zur Abschätzung von Quadraturfehlern für analytische Funktionen," *Numer. Math.*, v. 8, 1966, pp. 334–344. MR **34** #2179.