On the Calculation of the Inverse of the Error Function*

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Abstract. Formulas are given for computing the inverse of the error function to at least 18 significant decimal digits for all possible arguments up to $1\text{–}10^{-300}$ in magnitude.

A formula which yields erf (x) to at least 22 decimal places for $|x| \leqq 5\pi/2$ is also developed.

1. Introduction. In statistical work, many types of probability integrals or sums are approximated by functions which involve the normal probability integral or its inverse. Examples where the inverse is used in the asymptotic expansions of χ^2 distributions can be found in the first four references which are given at the end of this report. J. R. Philip [5] notes that the solution of a one-dimensional concentration-dependent diffusion equation can be obtained with the aid of the inverse error function, and also suggests some formulas which are useful for computation.

Formulas for the direct computation of the inverse error function have also been discussed by L. Carlitz [6]. Moreover, a computer program which obtains the inverse has recently been designed at the University of Chicago [7].

Throughout the remainder of this paper, we will use the notations

$$x = \text{erf}\,(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-t^2} dt \quad \text{and} \quad y = \text{inverf}\,(x)\,.$$

Since some formulas for y are obtained from numerical values of erf (y), it is necessary to consider the calculation of erf (y) also.

2. Formulas for erf(y). In the well-known Eq. [8]

$$\sum_{m=-\infty}^{\infty} \exp\,(-K(m+T)^2) = (\pi/K)^{1/2} \sum_{n=-\infty}^{\infty} \exp\,(-KT^2 + (KT + in\pi)^2/K)\,,$$

we take $K = 25\pi^2$ and $T \leqq \frac{1}{2}$ and obtain

$$e^{-(5\pi T)^2} + \epsilon(T) = \left[1 + 2\sum_{n=1}^{37} e^{-(n/5)^2} \cos 2n\pi T\right] / (5\sqrt{\pi})$$

where $|\epsilon(T)| < 10^{-25}$. If we take $5\pi T = z$ and integrate with respect to z from 0 to y, we see that

$$\text{(1)} \qquad \text{erf}\,(y) \approx \frac{2}{\pi}\left[y/5 + \sum_{n=1}^{37} n^{-1} e^{-(n/5)^2} \sin\,(2ny/5)\right] \quad \text{for} \quad |y| \leqq \frac{5\pi}{2}\,.$$

In order to circumvent the computation of the 37 values of sin $(2ny/5)$, we transform (1) essentially into a polynomial in $\alpha = 2C^2 - 1$, where $C = \cos\,(2y/5)$.

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From trigonometric identities, we have

$$\sin(2y(2n-1)/5) = S \cdot P_{2n-1} \text{ and } \sin(2y(2n/5)) = 2CS \cdot P_{2n}$$

where

$$S = \sin(2y/5), \qquad P_{m+1} = [1 + (1 + (-1)^m)(\tfrac{1}{2} + \alpha)]P_m - P_{m-1} \qquad (m \geqq 2)$$

with $P_1 = 1 = P_2$. When we substitute the appropriate $S \cdot P_{2n-1}$ and $2CS \cdot P_{2n}$ expressions into (1) and simplify the result, we obtain

$$\text{(2)} \qquad \operatorname{erf}(y) \approx 2y/(5\pi) + S \sum_{n=1}^{19} (A_{1n} + 2C \cdot A_{2n})\alpha^{n-1}.$$

The coefficients A_{1n} and A_{2n} are given in Table 1. These coefficients, as well as all the others given in this report, were computed on the CDC 3600 computer at Argonne National Laboratory.

Formula (2) was checked by comparing numerical values of erf (y) with the results of the series expansion

$$\operatorname{erf}(y) \approx \frac{2}{\sqrt{\pi}} y \sum_{n=0}^{25} \frac{(-y^2)^n}{n!(2n+1)}$$

for $y = 10^{-3}\ (10^{-3})\ 10^{-1}$. The maximum difference between corresponding values was never found to exceed 10^{-23} in magnitude.

For $y > 2$, we used the continued fraction [9]

$$\text{(3)} \qquad \int_y^\infty e^{-t^2} dt = \frac{e^{-y^2}}{2y+} \frac{1}{y+} \frac{2}{2y+} \frac{3}{y+} \frac{4}{2y+} \cdots$$

to obtain

$$\text{(4)} \qquad \operatorname{erf}(y) = 1 - \frac{2}{\sqrt{\pi}} \int_y^\infty e^{-t^2} dt.$$

The results of (2) and (4) were compared for $y = 2\ (.01)\ 7.85$, and again no differences between corresponding results were found to exceed 10^{-23} in magnitude.

3. The Calculation of inverf(x) **for Small** x. If primes indicate differentiation with respect to x, then from $x = \operatorname{erf}(y)$, we have $1 = (2/\sqrt{\pi})e^{-y^2}y'$, or

$$\text{(5)} \qquad y' = \frac{\sqrt{\pi}}{2} e^{y^2}.$$

Then

$$\text{(6)} \qquad y'' = 2yy'y'.$$

Carlitz [6] has developed a series expansion from a differential equation similar to (6). However, we will proceed in a different manner.

Equation (6) can be written as $y''(y')^{-2} = 2y$ and integrated to produce $-1/y' = 2 \int y\, dx + C$. From (5), it is evident that $y' = \sqrt{\pi}/2$ when $y = 0 = x$.

TABLE 1

Coefficients for calculating erf(Y) from formula (2)

n	A_{1n}					n	A_{2n}				
1	.70322	50027	43775	38882	17370	1	.24725	51681	40052	14690	13060
2	.33050	15219	16606	22314	14836	2	.14422	72263	61574	71714	84869
3	.20133	97472	64706	27322	06869	3	.08698	94549	95934	55384	75537
4	.10863	02450	22740	69068	73823	4	.04397	73381	94083	36808	21082
5	.04677	55234	32484	86082	31226	5	.01724	39625	88662	26211	67913
6	.01539	85726	15710	19966	35576	6	.00507	90696	12202	57028	12471
7	.00380	15076	79852	98711	51801	7	.00110	86064	53423	40653	70328
8	.00069	71837	92408	02873	57368	8	.00017	82280	16254	86169	73045
9	.00009	44909	26881	04549	80951	9	.00002	10404	58307	32513	81878
10	.00000	94328	11698	38366	78903	10	.00000	18206	63163	64340	76704
11	.00000	06919	27520	32514	00557	11	.00000	01153	30990	94436	94132
12	.00000	00372	25234	93691	07964	12	.00000	00053	42750	27603	08268
13	.00000	00014	66606	14233	80014	13	.00000	00001	80848	58780	95127
14	.00000	00000	42261	61443	18049	14	.00000	00000	04469	68229	24881
15	.00000	00000	00889	78652	67233	15	.00000	00000	00080	60688	38945
16	.00000	00000	00013	67604	44757	16	.00000	00000	00001	06013	64636
17	.00000	00000	00000	15334	23425	17	.00000	00000	00000	01016	49277
18	.00000	00000	00000	00125	36751	18	.00000	00000	00000	00007	10005
19	.00000	00000	00000	00000	74517	19	.00000	00000	00000	00000	00000

Consequently,

$$-1/y'(x) = 2\int_0^x y(t)dt - 2/\sqrt{\pi}\,. \tag{7}$$

Equation (7) can be used for analogue machine computation, since all values at $x = 0$ are known.

It may also be noted that if Eqs. (5) and (7) are combined, then

$$\int_0^x y(t)dt = (1 - e^{-y^2(x)})/\sqrt{\pi}\,.$$

A similar result which involves inverf $(1 - x)$ was obtained by Philip [5].

If we now assume that

$$\text{inverf}\,(x) = \sum_{n=1}^{\infty} C_n x^{2n-1} \tag{8}$$

for small x, then from (7)

$$1 + \left(\sum_{m=1}^{\infty} (2m-1)C_m x^{2m-2}\right)\left(\sum_{n=1}^{\infty} n^{-1}C_n x^{2n} - 2/\sqrt{\pi}\right) = 0\,. \tag{9}$$

The C_n values can be determined by multiplying the series of (9) and equating the coefficient of each power of x^2 to zero.

The first 200 values of C_n were computed and are given in Table 2. No attempt was made to determine the accuracy of these coefficients directly. Instead, Eq. (8) was used in the calculation of

$$\epsilon_1 = |x^{-1}\,\text{erf}\,(\text{inverf}\,(x)) - 1| \tag{10}$$

and

$$\epsilon_2 = |y^{-1}\,\text{inverf}\,(\text{erf}\,(y)) - 1| \tag{11}$$

for $x = .001\ (.001)\ .875$. In this range, the test calculations have not found any ϵ_1 or ϵ_2 as large as 10^{-22}.

Since the operations which produced Eq. (8) are also valid for complex values $x = z$, it should be possible to obtain good results from (8) whenever the inverse is unique. In this way, it should be feasible to obtain the inverse of Dawson's integral $\int_0^y e^{t^2}dt$ or other special functions for small arguments.

The first 200 terms of (8) were telescoped [10] by W. J. Cody, Jr. of Argonne for the range $|x| \leqq .8$. The result, equivalent in accuracy to (8), is expressed in the form

$$\text{inverf}\,(x) = x\left\{\xi_0 + \sum_{n=1}^{38} \xi_n T_n\left(\frac{x^2}{.32} - 1\right)\right\}, \tag{12}$$

where $T_n(\lambda)$ is the Chebyshev polynomial of degree n in λ and the ξ_n are the coefficients in Table 3.

4. Asymptotic Forms. Philip [5] suggests using a continued logarithm to obtain inverf (x) for large values of x. However, this asymptotic expansion appears to be accurate only for values of x which are very close to unity.

TABLE 2

Coefficients for the series expansion of the inverse, formula (8)

n	C_n					n	C_n				
1	.88622	69254	52758	01364	90836	69	.00318	86642	08655	62953	15348
2	.23201	36665	34654	49355	35340	70	.00313	89387	67365	57568	24422
3	.12755	61753	05597	95825	39997	71	.00309	06904	57324	04202	96607
4	.08655	21292	41547	53372	96417	72	.00304	38552	24041	35837	20698
5	.06495	96177	45385	41338	20146	73	.00299	83726	39899	61335	15649
6	.05173	12819	84616	37411	26318	74	.00295	41856	52066	84953	55223
7	.04283	67206	51797	34984	46514	75	.00291	12403	51090	82806	92453
8	.03646	59293	08531	62632	55797	76	.00286	94857	58223	83648	63406
9	.03168	90050	21605	44680	96094	77	.00282	88736	29736	73698	65527
10	.02798	06329	64995	22473	34306	78	.00278	93582	76662	85512	01777
11	.02502	22758	41198	34945	71692	79	.00275	08963	98573	52560	89963
12	.02260	98633	18897	57443	28165	80	.00271	34469	30129	83966	09447
13	.02060	67803	79059	00171	87687	81	.00267	69708	99281	70074	06481
14	.01891	82172	50778	85446	34987	82	.00264	14312	96097	70035	69962
15	.01747	63705	62856	54619	04295	83	.00260	67929	51309	30850	94100
16	.01623	15009	87685	25127	52949	84	.00257	30224	23741	94022	68714
17	.01514	63150	63247	80552	03853	85	.00254	00878	95884	91728	53698
18	.01419	23160	02509	96415	11535	86	.00250	79590	76923	32984	61222
19	.01334	73641	97421	29715	03471	87	.00247	66071	12618	30738	52077
20	.01259	40048	71332	06984	15966	88	.00244	60045	01479	09692	14999
21	.01191	82959	36392	03987	38582	89	.00241	61250	16721	37554	71575
22	.01130	89701	05922	53677	22250	90	.00238	69436	33552	05601	17534
23	.01075	68253	03317	95757	77905	91	.00235	84364	61362	05961	51350
24	.01025	42740	81853	46821	40955	92	.00233	05806	80445	64033	21513
25	.00979	50057	70071	17565	18125	93	.00230	33544	82898	29698	09601
26	.00937	37298	19182	08151	68329	94	.00227	67370	17375	50201	42631
27	.00898	59785	02843	37808	80376	95	.00225	07083	37421	75526	83530
28	.00862	79535	80709	43465	25409	96	.00222	52493	53104	14721	47496
29	.00829	64059	27739	24140	44119	97	.00220	03417	85706	96185	17663
30	.00798	85401	62603	35483	77731	98	.00217	59681	25264	10615	59824
31	.00770	19384	32259	76116	63687	99	.00215	21115	90724	53550	18068
32	.00743	44990	17831	53060	38419	100	.00212	87560	92562	54319	86507

33	.00718	43865	11268	31244	41163	101	.00210	58861	97659	96675	94849
34	.00694	99911	01064	71485	71204	102	.00208	34870	96301	18461	02249
35	.00672	98950	85341	52981	82763	103	.00206	15445	71134	36921	81473
36	.00652	28451	61450	06176	43492	104	.00204	00449	67963	94615	04750
37	.00632	77293	64343	14323	00182	105	.00201	89751	68249	69068	75947
38	.00614	35577	70384	15591	68944	106	.00199	83225	63197	47031	77333
39	.00596	94462	69734	45629	43731	107	.00197	80750	29335	39871	89774
40	.00580	46028	53834	30176	03007	108	.00195	82209	05477	18178	65060
41	.00564	83159	75551	56001	75138	109	.00193	87489	70981	77808	56061
42	.00549	99446	26203	96350	84038	110	.00191	96484	25225	22697	78422
43	.00535	89098	41686	44547	02891	111	.00190	09088	68206	67349	78892
44	.00522	46874	03687	49060	98435	112	.00188	25202	82216	29019	67351
45	.00509	68015	44706	20556	63718	113	.00186	44730	14498	00798	38444
46	.00497	48194	99739	04039	88281	114	.00184	67577	60844	76142	39462
47	.00485	83467	74958	48553	68178	115	.00182	93655	50068	36592	79292
48	.00474	70230	25884	97662	30607	116	.00181	22877	29290	20822	02917
49	.00464	05184	55559	03192	53643	117	.00179	55159	50002	67757	97464
50	.00453	85306	57907	08485	61814	118	.00177	90421	54854	72098	92192
51	.00444	07818	43527	18353	22720	119	.00176	28585	65118	09528	52763
52	.00434	70163	95021	51030	52998	120	.00174	69576	68793	83614	82027
53	.00425	69987	07182	72882	76970	121	.00173	13322	09321	18773	20596
54	.00417	05112	74126	17157	11220	122	.00171	59751	74853	75644	17219
55	.00408	73529	91109	02163	84896	123	.00170	08797	88069	98468	15080
56	.00400	73376	43498	14205	67945	124	.00168	60394	96487	20067	07082
57	.00393	02925	59306	47814	84431	125	.00167	14479	63250	50261	86294
58	.00385	60574	05048	21755	18009	126	.00165	70990	58369	59241	18616
59	.00378	44831	07473	82581	56007	127	.00164	29868	50378	39710	66003
60	.00371	54308	86126	04792	43302	128	.00162	91055	98393	91654	37246
61	.00364	87713	83679	09283	72093	129	.00161	54497	44552	22200	15137
62	.00358	43838	82744	56894	82347	130	.00160	20139	06800	91282	66518
63	.00352	21555	99297	86878	26452	131	.00158	87928	72028	62354	06210
64	.00346	19810	44137	66023	08552	132	.00157	57815	89513	37042	31735
65	.00340	37614	44872	07088	72919	133	.00156	29751	64672	64082	26346
66	.00334	74042	21855	56472	07579	134	.00155	03688	53099	16666	40669
67	.00329	28225	12303	31717	05625	135	.00153	79580	54867	29153	58463
68	.00323	99347	37504	23107	34137	136	.00152	57383	09095	74357	93174

TABLE 2—*Continued*

n	C_n					n	C_n				
137	.00151	37052	88753	46900	41999	169	.00120	68330	03777	02046	42050
138	.00150	18547	95695	96783	41014	170	.00119	91832	12277	46055	26983
139	.00149	01827	55920	30852	62930	171	.00119	16268	88071	40790	92689
140	.00147	86852	15027	68515	76807	172	.00118	41623	47882	70419	57329
141	.00146	73583	33883	02337	79937	173	.00117	67879	48370	71861	51896
142	.00145	61983	84461	74248	37436	174	.00116	95020	84957	43991	40250
143	.00144	52017	45874	34369	65163	175	.00116	23031	90695	55861	92051
144	.00143	43649	00560	02174	34877	176	.00115	51897	35175	88107	20751
145	.00142	36844	30640	99063	54024	177	.00114	81602	23472	49295	41168
146	.00141	31570	14429	67742	55385	178	.00114	12131	95124	16222	92518
147	.00140	27794	23081	37184	52451	179	.00113	43472	23150	53997	23961
148	.00139	25485	17385	32701	97047	180	.00112	75609	13101	78262	24858
149	.00138	24612	44687	68879	62726	181	.00112	08529	02140	28098	47558
150	.00137	25146	35939	99025	61333	182	.00111	42218	58153	23999	43965
151	.00136	27058	02867	28529	24407	183	.00110	76664	78894	90901	34486
152	.00135	30319	35250	31217	42029	184	.00110	11854	91157	31542	61606
153	.00134	34902	98316	37611	89685	185	.00109	47776	49968	40467	65333
154	.00133	40782	30233	92031	55990	186	.00108	84417	37816	53779	78759
155	.00132	47931	39706	01872	81788	187	.00108	21765	63900	34305	18471
156	.00131	56325	03658	27244	78368	188	.00107	59809	63402	96164	96955
157	.00130	65938	65016	82533	90847	189	.00106	98537	96789	76878	98422
158	.00129	76748	30572	43517	89698	190	.00106	37939	49128	69052	52188
159	.00128	88730	68926	74427	36023	191	.00105	78003	29432	27437	60287
160	.00128	01863	08517	08946	00913	192	.00105	18718	70020	70722	78881
161	.00127	16123	35716	37620	96728	193	.00104	60075	25905	00799	49419
162	.00126	31489	93004	71593	55317	194	.00104	02062	74189	65487	05342
163	.00125	47941	77209	69022	46255	195	.00103	44671	13493	93781	72699
164	.00124	65458	37812	26116	08073	196	.00102	87890	63391	35634	20867
165	.00123	84019	75315	49375	00053	197	.00102	31711	63866	41063	40786
166	.00123	03606	39673	39522	26384	198	.00101	76124	74788	16088	29286
167	.00122	24199	28777	30716	42512	199	.00101	21120	75399	95510	97017
168	.00121	45779	86997	41047	04374	200	.00100	66690	63824	75019	15111

TABLE 3

Coefficients for telescoped series, formula (12)

n	ξ_n					n	ξ_n				
0	.99288	53766	18940	82314	95800						
1	.12046	75161	43104	48646	47846	20	.00000	00000	00020	86508	97725
2	.01607	81993	42099	94472	57039	21	.00000	00000	00004	94880	41039
3	.00268	67044	37162	31582	79591	22	.00000	00000	00001	17663	94740
4	.00049	96347	30235	72629	47170	23	.00000	00000	00000	28038	55725
5	.00009	88982	18599	12044	09911	24	.00000	00000	00000	06695	06638
6	.00002	03918	12763	99443	37340	25	.00000	00000	00000	01601	65495
7	.00000	43272	71617	73542	18758	26	.00000	00000	00000	00383	82583
8	.00000	09380	81412	85934	06758	27	.00000	00000	00000	00092	12851
9	.00000	02067	34720	86834	27411	28	.00000	00000	00000	00022	14615
10	.00000	00461	59699	10543	00078	29	.00000	00000	00000	00005	33091
11	.00000	00104	16679	70271	46217	30	.00000	00000	00000	00001	28488
12	.00000	00023	71500	99959	21222	31	.00000	00000	00000	00000	31006
13	.00000	00005	43928	40684	71390	32	.00000	00000	00000	00000	07491
14	.00000	00001	25548	98640	97987	33	.00000	00000	00000	00000	01812
15	.00000	00000	29138	18036	63201	34	.00000	00000	00000	00000	00439
16	.00000	00000	06794	94218	08797	35	.00000	00000	00000	00000	00106
17	.00000	00000	01591	23433	31469	36	.00000	00000	00000	00000	00026
18	.00000	00000	00374	02505	85245	37	.00000	00000	00000	00000	00006
19	.00000	00000	00088	20877	62421	38	.00000	00000	00000	00000	00002

C. Hastings [11] essentially approximates the inverse by using rational functions of $(-\ln t^2)^{1/2}$ where $t = 1/(2\pi)^{1/2} \int_x^\infty e^{-z^2/2}dz$. Since these formulas are of limited accuracy, we recommend a slightly different form, which will now be justified.

Let

$$x^2 = (\text{erf}\, y)^2 = \frac{4}{\pi}\int_0^y e^{-s^2}ds \int_0^y e^{-t^2}dt = \frac{4}{\pi}\int_0^y \int_0^y e^{-(s^2+t^2)}dsdt\,.$$

The square over which the integration is performed can be decomposed into two regions, ψ_1 and ψ_2, where ψ_1 is the quarter circle $s^2 + t^2 \leqq y^2$, and ψ_2 is the remainder of the square. Converting to polar coordinates, we see that

$$\frac{4}{\pi}\int_{\psi_1} e^{-(s^2+t^2)}dsdt = \frac{4}{\pi}\int_0^{\pi/2}\int_0^y e^{-r^2}rdrd\theta = \int_0^y e^{-r^2}2rdr = 1 - e^{-y^2}\,.$$

Since

$$\frac{4}{\pi}\int_{\psi_2} e^{-(s^2+t^2)}dsdt < \frac{4}{\pi}\int_0^{\pi/2}\int_y^{y\sqrt{2}} e^{-r^2}rdrd\theta = e^{-y^2} - e^{-2y^2}\,,$$

this quantity can be neglected relative to $1 - e^{-y^2}$. Thus $x^2 \approx 1 - e^{-y^2}$ and we take $y \approx [-\ln(1 - x^2)]^{1/2}$ or

$$\text{inverf}\,(x) \approx (-\ln[(1 - x)(1 + x)])^{1/2} \qquad (13)$$

assuming positive x. Because of Eqs. (3) and (4) it is possible to preserve accuracy in $1 - x$.

To simplify notation, $\beta(x)$ will denote $[-\ln(1 - x^2)]^{1/2}$ throughout the remainder of this discussion.

Formula (13) can be improved if we define a new function $R(x)$ such that

$$\text{inverf}\,(x) = \beta(x)\cdot R(x)\,. \qquad (14)$$

For small x, $\beta(x)$ can be expanded in a power series. Because of this, a power series expansion was also generated for $R(x)$ making use of Eq. (8). The resulting series for $R(x)$ was found to be more strongly convergent than the series (8). Unfortunately, more effort is required to evaluate $\beta(x)$ than to compute the extra terms in (8).

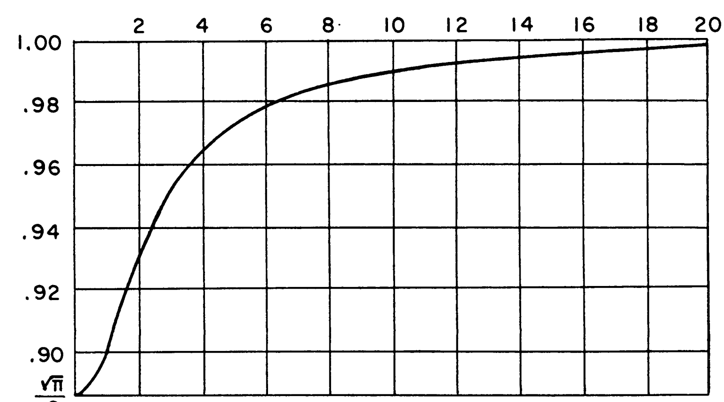

FIGURE 1. $R(X)$ VS. INVERF (X)

In Fig. 1 is a plot of $R(x)$ versus y. As the graph illustrates, $R(x)$ increases monotonically from $\sqrt{\pi}/2$ to 1 as y increases from 0 to ∞, showing that the relative error due to formula (13) is never larger than $2/\sqrt{\pi} - 1$.

The formulas for $R(x)$ which are given below were obtained by applying Chebyshev interpolation [12] to inverf $(x)/\beta(x)$.

For $.8 \leqq x \leqq .9975$,

$$R(x) \approx \sum_{n=0}^{26} \lambda_n T_n(D_1\beta(x) + D_2)\,, \tag{15}$$

where

$$D_1 = -1.54881\ 30423\ 73261\ 65951\ 2742\,,$$
$$D_2 = 2.56549\ 01231\ 47816\ 15192\ 8163\,,$$

and the coefficients λ_n are given in Table 4.

For $25 \cdot 10^{-4} \geqq 1 - x \geqq 5 \cdot 10^{-16}$,

$$R(x) \approx \sum_{n=0}^{37} \delta_n T_n(D_3\beta(x) + D_4)\,, \tag{16}$$

where

$$D_3 = -.55945\ 76313\ 29832\ 32254\ 36913\,,$$
$$D_4 = 2.28791\ 57162\ 63357\ 63896\ 5891\,,$$

and the coefficients δ_n are given in Table 5.

For $5 \cdot 10^{-16} \geqq 1 - x \geqq 10^{-300}$,

$$R(x) \approx \sum_{n=0}^{25} \mu_n T_n(D_5/(\beta(x))^{1/2} + D_6)\,, \tag{17}$$

where

$$D_5 = -9.19999\ 23588\ 30151\ 03127\ 8420\,,$$
$$D_6 = 2.79499\ 08201\ 24599\ 49376\ 8426\,,$$

and the coefficients μ_n are given in Table 6.

Considering the limitations of our formulas, function subroutines, and roundoff errors, these results are not as accurate as the length of the numbers given in Tables 4, 5, and 6 would seem to imply. Twenty-five decimals are given because it is not known which digits are significant.

Test cases which obtained ϵ_2 in (11) from equations (14) through (17) showed that $\epsilon_2 < 10^{-22}$.

A more severe test case using equations (3), (4), (14), (15), (16), and (17) which obtained

$$\epsilon_3 = |\lambda^{-1}[1 - \text{erf}\ (\text{inverf}\ (1 - \lambda))] - 1|$$

showed a larger error, with $\epsilon_3 < 10^{-19}$.

TABLE 4

Coefficients for calculating $R(x)$ from formula (15)

n	λ_n					n	λ_n				
0	.91215	88034	17553	77330	59200	14	.00000	00000	00052	53240	85874
1	−.01626	62818	67663	69585	46661	15	.00000	00000	00019	71154	08612
2	.00043	35564	72949	44536	50589	16	−.00000	00000	00001	74943	33828
3	.00021	44385	70074	45920	65205	17	−.00000	00000	00000	48005	96619
4	.00000	26257	51075	76481	30176	18	.00000	00000	00000	05573	02987
5	−.00000	30210	91050	10379	69912	19	.00000	00000	00000	01163	26054
6	−.00000	00124	06061	83675	72157	20	−.00000	00000	00000	00172	62489
7	.00000	00624	06609	29999	17380	21	−.00000	00000	00000	00027	84973
8	−.00000	00005	40124	79009	57858	22	.00000	00000	00000	00005	24481
9	−.00000	00014	23207	89753	15910	23	.00000	00000	00000	00000	65270
10	.00000	00000	34384	02819	55305	24	−.00000	00000	00000	00000	15707
11	.00000	00000	33584	87039	00138	25	−.00000	00000	00000	00000	01475
12	−.00000	00000	01458	42885	16512	26	.00000	00000	00000	00000	00450
13	−.00000	00000	00810	21742	58833						

TABLE 5

Coefficients for calculating $R(x)$ from formula (16)

n	δ_n					n	δ_n				
0	.95667	97090	20492	52745	26373	19	−.00000	00000	00964	84681	27965
1	−.02310	70043	09064	90369	99908	20	−.00000	00000	00219	56727	78128
2	−.00437	42360	97508	40773	33218	21	−.00000	00000	00009	56898	13014
3	−.00057	65034	22651	18548	09364	22	.00000	00000	00013	70325	72230
4	−.00001	09610	22307	09239	31242	23	.00000	00000	00006	25385	05417
5	.00002	51085	47024	64427	87982	24	.00000	00000	00001	45846	15266
6	.00001	05623	36067	94775	11955	25	.00000	00000	00000	10781	23993
7	.00000	27544	12330	03063	91503	26	−.00000	00000	00000	07092	29988
8	.00000	04324	84498	32833	80689	27	−.00000	00000	00000	03914	11775
9	−.00000	00205	30336	65520	86916	28	−.00000	00000	00000	01116	59209
10	−.00000	00438	91536	66543	16784	29	−.00000	00000	00000	00157	70366
11	−.00000	00176	84009	50808	81795	30	.00000	00000	00000	00028	53149
12	−.00000	00039	91289	02804	63420	31	.00000	00000	00000	00027	16662
13	−.00000	00001	86932	41245	59212	32	.00000	00000	00000	00009	57770
14	.00000	00002	72922	73967	46077	33	.00000	00000	00000	00001	76835
15	.00000	00001	32817	21315	65497	34	−.00000	00000	00000	00000	09828
16	.00000	00000	31834	24844	82286	35	−.00000	00000	00000	00000	20464
17	.00000	00000	01670	06077	51926	36	−.00000	00000	00000	00000	08020
18	−.00000	00000	02036	46496	11537	37	−.00000	00000	00000	00000	01650

TABLE 6

Coefficients for calculating $R(x)$ from formula (17)

n	μ_n					n	μ_n				
0	.98857	50640	66189	31364	60358	13	−.00000	00000	00000	35068	69329
1	.01085	77051	84599	47761	60281	14	−.00000	00000	00000	06972	21497
2	−.00175	11651	02762	79524	94825	15	−.00000	00000	00000	01095	67941
3	.00002	11969	93206	56334	37984	16	−.00000	00000	00000	00115	36390
4	.00001	56648	71404	24350	87911	17	−.00000	00000	00000	00013	26235
5	−.00000	05190	41686	91031	24261	18	−.00000	00000	00000	00002	63938
6	−.00000	00371	35789	74267	17780	19	.00000	00000	00000	00000	05341
7	.00000	00012	17430	86623	57429	20	−.00000	00000	00000	00000	22610
8	−.00000	00001	76811	55266	13442	21	.00000	00000	00000	00000	09552
9	−.00000	00000	11937	21825	56161	22	−.00000	00000	00000	00000	05250
10	.00000	00000	00380	25053	58299	23	.00000	00000	00000	00000	02487
11	−.00000	00000	00066	01883	22362	24	−.00000	00000	00000	00000	01134
12	−.00000	00000	00008	79170	55170	25	.00000	00000	00000	00000	00420

5. Comments on Errors. Since the result produced by the formulas of the preceding section includes an error, Dr. D. Woodward of Argonne contributed some of the ideas discussed below.

Let $y^* = y + \epsilon$ assuming $x = \operatorname{erf}(y)$ is exact. From Taylor's series and Eqs. (5) and (6) we obtain

$$\text{(18)} \quad \operatorname{inverf}(\operatorname{erf}(y^*)) = \operatorname{inverf}(\operatorname{erf}(y)) + \sqrt{\pi} e^{y^2} h_1/2 + \pi(y + \theta_1\epsilon)/4 \, (e^{2(y+\theta_1\epsilon)^2} h_1^2)$$

and

$$\text{(19)} \quad \begin{aligned} &\operatorname{inverfc}(\operatorname{erfc}(y^*)) \\ &= \operatorname{inverfc}(\operatorname{erfc}(y)) + \sqrt{\pi} e^{y^2} h_2/2 + \pi(y + \theta_2\epsilon)/4 \, (e^{2(y+\theta_2\epsilon)^2} h_2^2) \end{aligned}$$

where

$$h_1 = \operatorname{erf}(y^*) - \operatorname{erf}(y), \qquad 0 < \theta_1 < 1,$$

and

$$h_2 = \operatorname{erfc}(y^*) - \operatorname{erfc}(y), \qquad 0 < \theta_2 < 1.$$

If $\eta_m = \sqrt{\pi} e^{y^2} h_m/2$, $(m = 1, 2)$, then Eqs. (18) and (19) can be written as

$$\text{(20)} \qquad \epsilon = y^* - y = \eta_m + (y + \theta_m\epsilon) \exp\{2\theta_m\epsilon(2y + \theta_m\epsilon)\} \eta_m^2 .$$

Equation (20) shows that the error in y is approximately equal in magnitude to η_m when η_m is sufficiently small. For $y \leqq 2$, the computer program interpolated for y^* subject to the condition that $|h_1| = |\operatorname{erf}(y^*) - x| < x \cdot 10^{-23}$. Thus $|\eta_1| < \sqrt{\pi} e^{y^2}/2 \cdot x \cdot 10^{-23} < 10^{-21}$. This shows that it is possible to obtain y from x to at least 21 decimal places on the 3600 computer whenever $x \leqq \operatorname{erf}(2)$ is known to at least 24 significant decimal places.

For $y > 2$, y^* was obtained with the restriction that

$$|h_2| = |\operatorname{erfc}(y^*) - 1 + x| < (1 - x)\, 10^{-22} .$$

Then

$$|\eta_2| < \frac{\sqrt{\pi}}{2} e^{y^2} \left(\frac{2}{\sqrt{\pi}} \int_y^\infty e^{-t^2} dt 10^{-22} \right) < 10^{-22} .$$

Since y is never larger than 27 for the range under consideration, formula (20) implies that we can obtain y to at least 21 decimal places for $y > 2$ whenever $\operatorname{erfc}(y)$ is known to at least 22 significant figures.

Since y^* is assumed to be larger than .5, the relative error in y cannot be larger than 2ϵ.

6. Conclusion. Extensive testing with thousands of arguments of 24-decimal significance in the range $0 < |x| \leqq 1 - 10^{-300}$ and $0 < |y| \leqq 26.2$ showed that we should expect at least 18-decimal significance in the results of all formulas which were developed in this report.

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