Zeros of Sections of the Zeta Function. II

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1. Recapitulation. Paul Turán proved theorems connecting the locations of zeros of the Dirichlet polynomials

$$\zeta_N(s) = \sum_{n=1}^{N} n^{-s} \tag{1}$$

with the Riemann hypothesis. Let $s = \sigma + it$. One such theorem is that if all the zeros of every $\zeta_N(s)$ had real parts $\sigma \leqq 1$, then the Riemann hypothesis would be true. Unfortunately, this very simple condition, which could perhaps have been worked with in an inductive fashion, was shown by Haselgrove [1] to fail infinitely often. In part I of this paper (Spira [2]), a description was given of a calculation of zeros of $\zeta_N(s)$ for various N up to 10^{10}. No zero with $\sigma > 1$ was found.

In this concluding part, we apply in Section 2 generalizations of basic theorems of Bohr to $\zeta_N(s)$, and find g.l.b. $|\zeta_N(s)|$ for $\sigma \geqq 1$ and $N \leqq 5$. In Section 3 we discuss a confirmation of Haselgrove's proof, and report on related calculations. In Section 4, we describe machine proofs that $\zeta_N(s)$ has no zeros with $\sigma \geqq 1$ for $N \leqq 9$, and proofs of the existence of such zeros for a variety of small N starting with $N = 19$. Finally, we discuss our attempts at finding such zeros.

2. Applications of Bohr's Theorems. Let p_j be the jth prime, so $p_1 = 2$. For $n > 1$, let $r_{n,j}$ be the highest power of p_j dividing n, and let q_n be the index of the largest prime dividing n. We then can write (1) as

$$\zeta_N(s) = \sum_{n=1}^{N} n^{-\sigma} \exp\left(-it(r_{n,1}\log 2 + r_{n,2}\log 3 + \cdots + r_{n,q_n}\log p_{q_n})\right), \tag{2}$$

where we interpret the sum in parentheses as 0 when $n = 1$. Introducing the new variables $x_j = t \log p_j$, we now define the *companion function* of $\zeta_N(s)$:

$$F_N = F_N(\sigma, x_1, x_2, \cdots) = \sum_{n=1}^{N} n^{-\sigma} \exp\left(-i(r_{n,1}x_1 + r_{n,2}x_2 + \cdots + r_{n,q_n}x_{q_n})\right). \tag{3}$$

Since the $\log p_j$ are linearly independent over the rationals, and the $r_{n,j}$ are integers, we can apply generalizations (Spira [3]) of theorems of Bohr [4]. We can conclude first of all the set of values of $\zeta_N(s)$ for $\sigma \in (a, \infty)$ and $t \in (-\infty, \infty)$ is identical with the set of values of $F_N(\sigma, x_1, \cdots)$ where $\sigma \in (a, \infty)$ and each x_j runs independently over $[0, 2\pi)$. Thus, taking $a = 1$, $\zeta_N(s) = 0$ for some $\sigma > 1$ and some t if and only if $F_N(\sigma, x_1, \cdots) = 0$ for some (possibly different) $\sigma > 1$, and some values of the variables $x_1, x_2, \cdots$. We remark also that if $\zeta_N(s)$ has one zero with $\sigma > 1$, it has infinitely many (Spira [3]). Secondly, we can conclude that the values of

Received February 13, 1967.

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$F_N(\sigma, x_1, \cdots)$ where σ runs over a closed interval, and the x_j's run independently over $[0, 2\pi)$, form a closed set. Thus, for the σ-interval $[1, 2]$, the distance of this set to the origin is a well-defined constant, d_N.

Now, from the last two lines of p. 542 of I, we have that for $\sigma > 1$,

$$|\zeta_N(s)| \geqq 1 - 2^{-\sigma}(\sigma + 1)/(\sigma - 1) \tag{4}$$

and this last function increases with σ. At $\sigma = 2$, its value is 1/4, so for $\sigma \geqq 2$, $|\zeta_N(s)| \geqq 1/4$. Hence, for $\sigma > 2$, $|F_N| \geqq 1/4$, and if $d_N \leqq 1/4$, d_N is the minimum distance of F_N to the origin for all $\sigma \geqq 1$. For $5 \leqq N \leqq 50$ it turns out that $d_N \leqq 1/4$, but it is also true that for $N \leqq 4$, d_N is the minimum distance for all $\sigma \geqq 1$.

To discuss these $N \leqq 4$ cases, and for the sequel, we define:

$$\pi_N = \text{the set of primes } p \text{ satisfying } N/2 < p \leqq N \, , \tag{5}$$

$$\zeta_N^*(s) = \sum_{n=1; n \notin \pi_N}^{N} n^{-s} \, , \qquad \pi_N(s) = \sum_{n \in \pi_N} n^{-s} \, , \tag{6}$$

$$F_N^* = F_N^*(\sigma, x_1, \cdots) = \text{the companion function of } \zeta_N^*(s) \, , \tag{7}$$

$$P_N = P_N(\sigma_1, x_1, \cdots) = \text{the companion function of } \pi_N(s) \, . \tag{8}$$

We have $\zeta_N(s) = \zeta_N^*(s) + \pi_N(s)$, $F_N = F_N^* + P_N$. Note that σ is the only variable F_N^* and P_N have in common. The first four $\zeta_N^*(s)$ are 1, 1, 1, $1 + 2^{-s} + 4^{-s}$. In general, we have

$$\begin{aligned} \zeta_N^*(s) &= \zeta_{N-1}^*(s) \text{ if } N \text{ is a prime} \, , \\ \zeta_N^*(s) &= \zeta_{N-1}^*(s) + (N/2)^{-s} + N^{-s} \text{ if } N \text{ is twice a prime} \, , \\ \zeta_N^*(s) &= \zeta_{N-1}^*(s) + N^{-s} \text{ otherwise} \, , \end{aligned} \tag{9}$$

and there are similar equations for F_N^*. It is clear that the variables in P_N can be chosen so that P_N is a vector in any assigned direction of length $\sum_{p \in \pi_N} p^{-\sigma}$. We avoid the general question of whether values of $\sigma, x_1, \cdots$ which minimize $|F_N^*|$ also minimize $|F_N|$ after suitable selection of the variables in P_N. For $N \leqq 5$, this turns out to be true, since the minima for $\sigma \geqq 1$ of $|F_N^*|$ lie at the extreme $\sigma = 1$. The remarks above on the relations of the sets of values of $\zeta_N(s)$ and F_N also carry over to the functions $\zeta_N^*(s)$ and F_N^*.

For $N = 1$, it is trivial that $d_N = 1$. A short calculation shows that for $N = 2$, we obtain a minimum $d_2 = 1/2$ at $\sigma = 1$, $x_1 = \pi$, and for $N = 3$, $d_3 = 1/6$ and is attained at $\sigma = 1$, $x_1 = x_2 = \pi$. It is also easy to see that these three minima hold for $\sigma \geqq 1$. We sketch the calculation of d_4.

We consider first F_4^*. We have $|F_4^*| \geqq 1 - 2^{-\sigma} - 4^{-\sigma}$, which is > 0 for $\sigma \geqq 1$. Thus, for some σ in $[1, 3]$ and for some x_1 in $[0, 2\pi)$, $|F_4^*|$ takes on a positive minimum. In finding such σ and x_1, we can study $g(\sigma, x_1) = |F_4^*|^2$ instead of $|F_4^*|$. At a minimum, we must have $\partial g/\partial x_1 = 0$, (since g is periodic of period 2π in x_1, we do not have to consider extreme values in that variable). A short calculation gives

$$g(\sigma, x_1) = 1 + 2^{-2\sigma} + 4^{-2\sigma} + 2^{1-\sigma}[1 + 4^{-\sigma}] \cos x_1 + 2 \cdot 4^{-\sigma} \cos 2x_1 \, , \tag{10}$$

$$\partial g/\partial x_1 = -2^{1-\sigma}(\sin x_1)[1 + 4^{-\sigma} + 2^{2-\sigma} \cos x_1] \, . \tag{11}$$

Thus, $\partial g/\partial x_1 = 0$ if and only if $x_1 = 0$ or π, or $\cos x_1 = -(2^\sigma + 2^{-\sigma})/4$. For $x_1 = \pi$, setting $x = 2^{-\sigma}$, we obtain $g(\sigma, \pi) = (x^2 - x + 1)^2$. It is easily seen that $x^2 - x + 1 > 0$, and has its minimum at $x = 1/2$ or $\sigma = 1$, where $|F_4^*| = 3/4$.

For $x_1 = 0$, we obtain $g(\sigma, 0) = (x^2 + x + 1)^2$, which is greater than $(x^2 - x + 1)^2$ $(= g(\sigma, \pi))$ if $x > 0$, which we can assume as $x = 2^{-\sigma}$. Thus, the minimum cannot be attained for $x_1 = 0$.

In the final case, using $\cos 2x_1 = 2\cos^2 x_1 - 1$, if $\cos x_1 = -(2^\sigma + 2^{-\sigma})/4$, we obtain $\cos 2x_1 = (4^\sigma - 6 + 4^{-\sigma})/8$ and $g(\sigma, x_1) = \frac{3}{4}(1 - 4^{-\sigma})^2$, which is least at $\sigma = 1$, and indeed gives the least possible $|F_4^*|$ of $3\sqrt{3}/8$, at $\cos x_1 = -5/8$. This gives $d_4 = (9\sqrt{3} - 8)/24$ and $d_5 = (45\sqrt{3} - 64)/120$, where we choose x_2 and x_3 so that the vectors $\exp(-ix_2)$ and $\exp(-ix_3)$ point opposite to $F_4^*(1, \arccos(-5/8))$.

If $F_N(\sigma, x_1, \cdots) = 0$, and if the appropriate Jacobian does not vanish, we can solve the equation for σ, and thus obtain a σ interval in which F_N vanishes. The zeros of $\zeta_N(s)$ will have real parts dense in this interval. The empirical results suggest the conjecture that to each $\zeta_N(s)$ there is a single such interval, though it could possibly arise from overlapping F_N σ-intervals.

If we take F as the companion function of a general Dirichlet series, it is not clear what we should do about such solvability conditions, since F will have infinitely many variables.

3. Calculations Related to Haselgrove's. The Dirichlet polynomial

$$L_N(s) = \sum_{n=1}^{N} \frac{\lambda(n)}{n^s} \tag{12}$$

where $\lambda(n)$ is Liouville's function, is equivalent in the sense of Bohr [4] to $\zeta_N(s)$, and hence assumes the same set of values in any half plane $\sigma > \sigma_0$. If s is real and large, then $L_N(s)$ is near 1. Thus, if $L_N(1) < 0$, then there would be a real root of $L_N(s)$ larger than 1, and hence also a root of $\zeta_N(s)$ with $\sigma > 1$.

The author found that $L_N(1) > 0$ for $N \leqq 824{,}000$, and found $L_{293}(1) = .0051122775$, $L_{1000}(1) = .0289948068$, values slightly different from those appearing in Turán [6]. The lowest value obtained was $L_{96862} = .00011996$. R. Sherman Lehman's [5] values of $L_N(0)$ for $N = 200{,}000(200{,}000)800{,}000$ were verified.

To study $L_N(1)$ further, one may use analytic expressions, derived by the calculus of residues (Haselgrove [1], Lehman [5]). Indeed, setting

$$L_1(x) = \sum_{n \leqq x} \frac{\lambda(n)}{n} \tag{13}$$

the expression

$$B_T(u) = \frac{-1}{\zeta(\frac{1}{2})} + \sum_{|\gamma_n| < T} \frac{\zeta(2\rho_n)}{(\rho_n - 1)\zeta'(\rho_n)} \exp(i\gamma_n u), \tag{14}$$

where $\rho_n = \frac{1}{2} + i\gamma_n$ are roots of $\zeta(s)$, under various unproved hypotheses, can be shown to represent $e^{u/2} L_1(e^u)$, with some blurring. The focusing improves as T increases. For example, for $T = 200$ and $u \leqq 2$, one can readily see rather sharp changes (without a Gibbs effect) as $L_1(x)$ makes a step. Lehman [5] used a function similar to (14) to successfully guess where $L(x) = \sum_{n \leqq x} \lambda(n)$ changed sign.

An expression the same as (14) but with the factor $(1 - \gamma_n/T)$ inside the sum was used by Haselgrove to show that $L_1(x)$ is negative infinitely often. We designate this sum by $B_T^*(u)$. For corresponding sums for $L(x)$ we use the notation $A_T(u)$ and $A_T^*(u)$, as used by Lehman [5]. Finally, by $C_T(u)$ and $C_T^*(u)$ we mean the corresponding sums for the function $M(x) = \sum_{n \leqq x} \mu(n)$, where $\mu(n)$ is the Möbius function. We have

$$C_T(u) = \sum_{|\gamma_n|<T} \frac{\exp(i\gamma_n u)}{\rho_n \zeta'(\rho_n)}, \tag{15}$$

which represents, hopefully, $e^{-u/2}M(e^u)$. Formulas for $A_T(u)$ and $A_T^*(u)$ can be found in Lehman [5], and for $B_T^*(u)$ and $C_T^*(u)$ in Haselgrove [1].

All six of these functions were calculated in double precision for $T = 100$, $u = 0(.01)500$, and for $T = 200$, 500, and 1000 in selected ranges. The coefficients for these functions were calculated in double precision, calculating first improved γ_n from the 6D values in Haselgrove and Miller [7]. We first discuss the tables in Haselgrove [1]. Write $\alpha_n = \zeta(2\rho_n)/\rho_n\zeta'(\rho_n)$. In Table I, the γ_n are correct to within 3 units in the 10th significant figure. For the first six $|\alpha_n|$, terminal digits 19,8,5,3,2,993, were obtained rather than 23,6,8,5,4,878. For $n = 9,15,18,34,48$ the terminal digits 4,4,6,6,3 were obtained rather than 5,3,5,7,4. The quantity $(ph\ \alpha_n)/\pi$ was not checked. The values of $A_{1000}^*(u)$ in Table II were confirmed within 1 unit except for the five values starting with 831.837, which are two units low. Also, the value at 831.857 was found to be $-.06320$. In Lehman's [5] paper the value $A_{1000}(814.492)$ was found to have terminal digit 0, and $A_{1000}(831.847)$ was found to be .0049448.

Two new places were found where $A_T^*(u) > 0$: $A_{1000}^*(310.8276) = .0109$, $A_{1000}^*(384.690) = .0316$. High maxima also occur at $u = 33.495$, 44.591 and 214.404.

For $L_1(x)$, the author found that $B_{1000}^*(853.853) = -.0321$ and $B_{1000}^*(996.980) = -.0450$ and $B_{1000}^*(996.981) = -.0457$, confirming Haselgrove's proof. It was also found that $B_{1000}^*(171.4938) = -.0009$ and $B_{1000}^*(331.9602) = -.0170$, giving two new places where this function is negative. Low minima occur at $u = 43.897$, 54.624, 124.843, 188.830, and 437.758.

To disprove Mertens hypothesis it would be sufficient to find u and T such that $|C_T^*(u)| > 1$. No such values were found. Table I gives places where $|C_{1000}^*(u)|$ rises above .5.

TABLE I. *Approximate Values for* $e^{-u/2}M(e^u)$

u	$C_{1000}^*(u)$	u	$C_{1000}^*(u)$
22.7730	+.5003	441.5100	+.5145
43.8965	−.5199	480.6430	+.5069
97.5260	+.5355	814.4910	+.5061
310.8258	+.5301	853.852	−.6027

4. Machine proofs. We first describe the proofs that $\zeta_N(s) \neq 0$ for $\sigma \geqq 1$, $N \leqq 9$. The idea of such a proof is, for one variable, based on the formulas

$$|f(x_0 + h)| \geqq |f(x_0)| - \max |h| \cdot \max |f'(\xi)|, \tag{16}$$

$$(17)\quad |f(x_0+h)| \geqq |f(x_0)| - \max|h|\ |f'(x_0)| - (\max|h|^2/2!)\max|f''(\xi)|\,,$$

which are easily derivable from the Taylor's expansion under suitable restrictions on f. Thus, from formula (16), if $|f'(\xi)|$ is suitably bounded, and $f(x_0) \neq 0$, we can conclude that $f(x) \neq 0$ for a small interval about x_0. Formula (17) is useful when $|f(x)|$ has a small minimum, as in the cases we consider. We then get some help from $|f'(x_0)|$ being small near the minimum, and from the $|h|^2$ in the next term. These formulas easily generalize to the case of f being a real or complex function of several real variables. Roundoff also must be taken into account.

In our particular case, we can take advantage of the special nature of our functions F_N, and consider only the variables appearing in $F_N{}^*$. If we take the variables $\sigma, x_1, \cdots$, to lie in a cube C, we have at any point in C,

$$(18)\qquad |F_N| \geqq |F_N{}^*| - |P_N| \geqq \operatorname*{g.l.b.}_C |F_N{}^*| - \operatorname*{l.u.b.}_C |P_N|\,.$$

Thus, $|F_N| > 0$ in C provided $\text{g.l.b.}_C\,|F_N{}^*| > \text{l.u b.}_C\,|P_N|$ $(=\sum_{p\in\pi_N} p^{-\min\sigma})$. Now we can apply formulas of the type (16) or (17). Write $F_N{}^* = u + iv$. A formula corresponding to (16) is

$$(19)\qquad \begin{aligned} &|F_N{}^*(\sigma+h, x_1+h_1, \cdots)| \\ &\quad \geqq |F_N{}^*(\sigma, x_1, \cdots)| - \max|h|\left[\max\left|\frac{\partial u}{\partial\sigma}\right| + \max\left|\frac{\partial v}{\partial\sigma}\right|\right] \\ &\qquad - \max|h_j|\left[\sum_i\left(\max\left|\frac{\partial u}{\partial x_i}\right| + \max\left|\frac{\partial v}{\partial x_i}\right|\right)\right], \end{aligned}$$

and one corresponding to (17) is

$$(20)\qquad \begin{aligned} |F_N{}^*(\sigma+h, x_1+h_1, \cdots)| \geqq\ &|F_N{}^*(\sigma, x_1, \cdots)| \\ &- \max|h|\left[\left|\frac{\partial u}{\partial\sigma}(\sigma, x_1, \cdots) + i\frac{\partial v}{\partial\sigma}(\sigma, x_1, \cdots)\right|\right] \\ &- \max|h_j|\left[\sum_i\left|\frac{\partial u}{\partial x_i}(\sigma, x_1, \cdots) + i\frac{\partial v}{\partial x_i}(\sigma, x_1, \cdots)\right|\right] \\ &- \frac{\max|h|^2}{2!}\left[\max\left|\frac{\partial^2 u}{\partial\sigma^2}\right| + \max\left|\frac{\partial^2 v}{\partial\sigma^2}\right|\right] \\ &- \frac{\max|h_j{}^2|}{2!}\left[\sum_{i,j}\left(\max\left|\frac{\partial^2 u}{\partial x_i\partial x_j}\right| + \max\left|\frac{\partial^2 v}{\partial x_i\partial x_j}\right|\right)\right] \\ &- \max|hh_j|\left[\sum_i\left(\max\left|\frac{\partial^2 u}{\partial\sigma\partial x_i}\right| + \max\left|\frac{\partial^2 v}{\partial\sigma\partial x_i}\right|\right)\right]. \end{aligned}$$

It was not possible to avoid using (20). The expressions in (20) can be simplified. We have, writing $\pi_N{}^*$ as $[1, N] - \pi_N$,

$$(21)\qquad \max\left|\frac{\partial u}{\partial\sigma}\right| + \max\left|\frac{\partial v}{\partial\sigma}\right| \leqq 2\sum_{n\in\pi_N{}^*}(\log n)n^{-\min\sigma}$$

and using the notation of (2),

$$(22)\quad \sum_i \left(\max \left| \frac{\partial u}{\partial x_i} \right| + \max \left| \frac{\partial v}{\partial x_i} \right| \right) \leqq 2 \sum_{n \in \pi N^*} (r_{n,1} + r_{n,2} + \cdots + r_{n,q_n}) n^{-\min \sigma}.$$

For formula (20) we can use

$$(23)\quad \max \left| \frac{\partial^2 u}{\partial \sigma^2} \right| + \max \left| \frac{\partial^2 v}{\partial \sigma^2} \right| \leqq 2 \sum_{n \in \pi N^*} (\log n)^2 n^{-\min \sigma},$$

$$(24)\quad \sum_{i,j} \left(\max \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right| + \max \left| \frac{\partial^2 v}{\partial x_i \partial x_j} \right| \right) \leqq 2 \sum_{n \in \pi N^*} (r_{n,1} + \cdots + r_{n,qn})^2 n^{-\min \sigma},$$

$$(25)\quad \sum_i \left(\max \left| \frac{\partial^2 u}{\partial \sigma \partial x_i} \right| + \max \left| \frac{\partial^2 v}{\partial \sigma \partial x_i} \right| \right) \leqq 2 \sum_{n \in \pi N^*} (\log n)(r_{n,1} + \cdots + r_{n,q_n}) n^{-\min \sigma}.$$

To save computation, the machine proof was attempted simultaneously for those N's for which the $F_N{}^*$ have the same number of variables x_j. The process of proof starts with the cube σ: [1, 2], x_j: $[0, 2\pi]$, $j = 1, \cdots$. One could limit one of the x_j's to $[0, \pi]$, since $|F_N(\sigma, x_1, x_2, \cdots)| = |F_N(\sigma, 2\pi - x_1, 2\pi - x_2, \cdots)|$. One breaks the cube into smaller cubes, and checks by (19) and then (20) to see if $|F_N| > 0$ throughout each smaller cube. The smaller cubes not satisfying this are further refined. The final program used integer pair coordinates for the x_j's, $(m, n) = 2m\pi/n$, where n was chosen a power of 2. All the cubes of a size were examined together, so that the sines and cosines could be computed just once for a given set of cubes. Also, the coordinates of min $|F_N|$ were saved. If one attempts to use condition (19) alone, the number of cubes rises to an impractical level.

Table II gives results of the proofs for $N = 6$ to 9 at several stages. The roundoff leeway was taken as 10^{-5}. Column 1 contains the σ-width of the cubes. Column 2 has the number of division of 2π for the x_j edge length. Column 3 gives the σ-coordinate of the center of the cube with minimum $|F_N|$, which turned out to be the same for $N = 6$ to 9. The next three columns give min $|F_6|$ and the integer first coordinates of x_1 and x_2 where this minimum was attained. The second coordinate is twice the value in column 3, (since we are calculating at the *centers* of cubes, the program needs a mesh half the edge). For example, in the first row, we are considering cubes of σ-width .25, and x_j width $2\pi/16$. The min $|F_6|$ is .28384 and is attained at $x_1 = 11 \cdot (2\pi/32)$, $x_2 = 15 \cdot (2\pi/32)$. The next nine columns give corresponding data for $N = 7, 8, 9$. The last two columns give the letters Y and N according to whether a proof was obtained or not. The letters are in order corresponding to $N = 6$ to 9. The first of these columns gives the proof results obtained using (19) above, and the second, the results obtained using (20) also. The results in the $N = 8$ columns indicate that the proof first sought out a secondary minimum, which was later calculated, and then finally found a cube where there was a lower minimum as the mesh refined. Each set of cubes was processed completely to find min $|F_N|$, rejecting when possible, using the current min $|F_N|$, and then a second pass made using the final values of min $|F_N|$ to reject further cubes. The program also saved cubes where there was a possibility of lower min $|F_N|$ within the cube. The total run time was less than 20 minutes.

The min $|F_N|$ in the table were recalculated in double precision, and a separate confirmatory calculation was performed along $\sigma = 1$ which found x_1 and x_2 which minimized $|F_N(1, x_1, x_2)|$ for a mesh of $2\pi/650$. Values of x_1, x_2 which gave $|F_N(1, x_1, x_2)|$ slightly greater than the minimum were also saved, and studied, and

TABLE II
Proof Data, $N = 6, 7, 8, 9$

σ mesh	x mesh	σ min	$N = 6$			$N = 7$			$N = 8$			$N = 9$			Pr. 1	Pr. 2
.250	16	1.125	.28384	11	15	.17182	11	15	.23154	15	15	.25615	15	11	*NNNN*	*NNNN*
.125	32	1.0625	.23256	21	31	.10607	21	31	.17370	31	31	.20056	31	21	*NNNN*	*YNYY*
.0625	64	1.03125	.20289	43	61	.06846	43	61	.14455	63	63	.16898	61	43	*YNNN*	*YYYY*
.03125	128	1.015625	.18778	85	121	.04920	85	121	.12941	71	123	.15287	121	85	*YNYY*	*YYYY*

TABLE III

Local Minima of $|F_N^*|$ *on* $\sigma = 1$ *and Angles where Attained*

N	local minimum	less $\sum_{p \in \pi_N} p^{-1}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
1	1	+ 1									
2	1	+ 1/2									
3	1	+ 1/6									
4	$3\sqrt{3}/8$	+ .316186*	2.245928†								
5		+ .116186*									
6	.372759	+ .172759	2.091870	2.953272							
7		+ .029902									
8	.452582	+ .109724	1.721394	3.044474							
9	.479230	+ .136372	2.986152	2.062967							
10	.286430	+ .143573	1.739896	2.496772	3.168891						
11		+ .052664									
12	.330873	+ .097107	1.633311	2.376010	3.218477						
13		+ .020184									
14	.173630	+ .005798	1.586010	2.404622	3.250068	3.250068					
15	.183411	+ .015578	1.626395	2.278784	2.974448	3.283475					
16	.232361	+ .064529	1.471382	2.331302	3.030376	3.321327					

17		+ .005705									
18	.264527	+ .037871	1.462765	2.159240	3.059133	3.346347					
19		− .014760									
20	.287264	+ .007976	1.397744	2.199895	2.891836	3.378705					
21	.287901	+ .008614	1.429136	2.110608	2.917566	3.114078					
22	.180110	− .008267	1.399971	2.127013	2.932031	3.131736	3.411964				
23		− .051746									
24	.210415	− .021442	1.341302	2.094990	2.956244	3.156690	3.430748				
25	.216213	− .015643	1.369312	2.126559	2.614426	3.174392	3.452011				
26	.1232	− .0317	1.3466	2.1355	2.6293	3.1880	3.4641	3.4641			
27	.1472	− .0078	1.3704	1.9584	2.6683	3.2073	3.4750	3.4750			
28	.1595	+ .0046	1.3294	1.9880	2.6900	3.0328	3.4947	3.4947			
29		− .0299									
30	.1741	− .0153	1.3277	1.9442	2.5888	3.0524	3.5086	3.5086			
31		− .0476									
32	.2003	− .0214	1.2591	1.9723	2.6129	3.0663	3.5224	3.5224			
33	.1957	− .0260	1.2812	1.9135	2.6242	3.0796	3.2710	3.5297			
34	.1225	− .0403	1.2609	1.9239	2.6351	3.0879	3.2797	3.5382	3.5382		
35	.1249	− .0379	1.2818	1.9417	2.5294	2.9352	3.2905	3.5516	3.5515		

* $3\sqrt{3}/8$ less 1/3 and then less 1/5. † arc cos (−5/8).

TABLE III (*Continued*)

36	.1457	− .0171	1.2611	1.8813	2.5547	2.9571	3.3053	3.5597	3.5592		
37		− .0441									
38	.0798	− .0573	1.2463	1.8848	2.5610	2.9645	3.3136	3.5698	3.5716	3.5702	
39	.0743	− .0629	1.2608	1.8496	2.5738	2.9724	3.3166	3.3138	3.5631	3.5669	
40	.0910	− .0462	1.2219	1.8684	2.5191	2.9883	3.3300	3.3288	3.5819	3.5822	
41		− .0706									
42	.1018	− .0599	1.2195	1.8416	2.5328	2.8826	3.3419	3.3426	3.5926	3.5940	
43		− .0831									
44	.1044	− .0805	1.1909	1.8551	2.5440	2.8990	3.1698	3.3567	3.6064	3.6061	
45	.1173	− .0675	1.2064	1.8015	2.4921	2.9096	3.1784	3.3622	3.6074	3.6076	
46	.0622	− .0792	1.2010	1.8018	2.4845	2.9079	3.1822	3.3727	3.6229	3.6238	3.6259
47		− .0805									
48	.0793	− .0834	1.1624	1.7952	2.5120	2.9342	3.1987	3.3798	3.6215	3.6205	3.6191
49	.0808	− .0819	1.1836	1.8184	2.5186	2.7110	3.1941	3.3758	3.6193	3.6229	3.6235
50	.0906	− .0721	1.1840	1.8268	2.3905	2.7145	3.2001	3.3875	3.6373	3.6392	3.6405

other checking computations were performed.

It would not be difficult now to repeat the computational proof.

Minima of $|F_{10}^*|$ and $|F_{12}^*|$ were also sought with a mesh of $2\pi/150$, and of $|F_{14}^*|$ through $|F_{21}^*|$ with a mesh of $2\pi/50$. Local minima were then sought by a minimum search program, using starting values obtained from these initial searches. The searching program simply stepped each variable by a quantity h, halving h, for a number of times, as the minimum was found with mesh h. For $N > 21$, further minima were also sought, using as starting values the quantities x_j at which $|F_{N-1}^*|$ was a minimum, taking π for the initial value of any new variable.

Table III gives the results of these computations. The x_j are given in radians. The quantities for $N \leqq 35$ should be accurate to one unit in the last place, as they were computed in double precision. For $35 < N \leqq 50$, the quantities should be correct to within 3 units in the last place. It is not claimed that the local minima are the true absolute minima of $|F_N^*|$.

Since $|F_N^*| \to 1$ and $\sum_{p \in \pi_N} p^{-\sigma} \to 0$ as $\sigma \to \infty$, if we find values of the variables $x_1, \cdots$, so that $|F_N^*(1, x_1, \cdots)| - \sum_{p \in \pi_N} 1/p < 0$, then $F_N(\sigma, x_1, \cdots)$ has a root with $\sigma > 1$. Thus, from Table III, for $N = 19$, 23 to 27 and 29 to 50, F_N has roots with $\sigma > 1$.

It is possible that such searches for minima could be speeded by using a gradient method. If this were so, one could write a general program for computing successive minima of the $|F_N^*|$ and study whether this situation of zeros for $\zeta_N(s)$ for $N \geqq 29$ continues to hold. We remark that it follows from Rosser and Schoenfeld's [8] inequalities for $\sum_{p \leqq x} 1/p$ that $\sum_{p \in \pi_N} 1/p$ is approximately $\log 2/\log N$ and tends to zero as $N \to \infty$.

To find a zero of $\zeta_{19}(s)$ with $\sigma > 1$, one can seek t for which $t \log p_j \equiv x_j + \epsilon_j \pmod{2\pi}$, $j = 1, \cdots, 4$, where the x_j have the values of Table III for $N = 19$ and ϵ_j is small. For $j = 1$, we can make $\epsilon_1 = 0$ by choosing $t = (x_1 + 2k\pi)/\log 2$, $k = 0, \pm 1, \pm 2, \cdots$. Then for each k we can check if $t \log p_j \equiv x_j \pmod{2\pi}$ within ϵ_j, for $j = 2, 3, 4$, where we preassign the ϵ_j's. If one accumulated a sufficiently large number of such cases, and if 11^{-s}, 13^{-s}, 17^{-s} and 19^{-s} are randomly pointed, one could hope to find a case of a zero of $\zeta_{19}(s)$ beyond $\sigma = 1$. Efforts along these lines produced the zero of $\zeta_{23}(s)$ $.9705 + i\, 10716133.0062$, which has real part somewhat beyond that of the lowest zero $.9325 + i\, 1.6975$.

The calculations were performed on an IBM 7040 at the University of Tennessee Computing Center, which was aided by grants NSF-G13581 and NSF-GP4046.

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