Amicable Numbers and the Bilinear Diophantine Equation*

By Elvin J. Lee

1. The Bilinear Diophantine Equation. The integer solution of

$$axy + bx + cy = d \tag{1}$$

may be reduced to a factorization. Multiplying (1) by a, adding bc to both sides and factoring results in

$$(ax + c)(ay + b) = ad + bc\,. \tag{2}$$

If n is a factor of $ad + bc$ and a divides $n - c$, the integer solution of (1) is

$$x = (n - c)/a\,, \quad y = (m - b)/a \quad \text{where} \quad mn = ad + bc\,. \tag{3}$$

2. Amicable Numbers. *Method* I. An amicable pair, (n_1, n_2) is defined by

$$\sigma(n_1) = \sigma(n_2) = n_1 + n_2 \tag{4}$$

where σ denotes the divisor sum function [1]. If we let

$$n_1 = Apq \quad \text{and} \quad n_2 = Br \tag{5}$$

where p, q are primes relatively prime to some number A, and r is a prime relatively prime to a number B, (4) gives

$$S(p + 1)(q + 1) = T(r + 1) = Apq + Br \quad \text{where } S = \sigma(A)\,, \quad T = \sigma(B)\,. \tag{6}$$

Solving for r and eliminating r gives

$$r = S(p + 1)(q + 1)/T - 1 \tag{7}$$

and

$$[AT - S(T - B)]pq - S(T - B)(p + q) = BT + S(T - B)\,. \tag{8}$$

Substituting in (2) we obtain

$$ad + bc = T[ABT + (A - B)(T - B)S]\,. \tag{9}$$

Our procedure now is to find all factor pairs, M, N, of $T[ABT + (A - B)(T - B)S]$ such that $MN = T[ABT + (A - B)(T - B)S]$. For any such pair we solve the linear equations,

$$[AT - S(T - B)]p = N + S(T - B)\,, \tag{10a}$$

$$[AT - S(T - B)]q = M + S(T - B) \tag{10b}$$

given by (3) observing that (10b) has a solution if and only if (10a) has a solution

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and requiring $N < M$ to avoid duplication in view of p, q symmetry of (8). All p, q thus found are checked for primality by a sieve algorithm. When p and q are both primes, r is computed by (7) and checked for primality.

The procedure is programmed in fixed-point arithmetic in order to find quickly the rather small number of integer solutions of (7) to be checked for primality. A and B are initially selected such that $A/S + B/T > 1$.

Limitations on fixed-point word size (47 bits for the 1604) made a systematic and exhaustive search impossible. The procedure was also programmed in floating-point arithmetic (75 bits), but slow running speed limited usefulness of this version. Another fixed-point version utilizing a preliminary extraction of common factors at appropriate stages of the calculations seems promising in extending the search to larger numbers, especially since faster machines are already available and still faster machines will be in the near future.

Method II. This method is less general than the first method described herein but may be of use when computer word-size limitations vitiate Method I. Its general plan is to eliminate a suitable variable from a specific form of Eqs. (4) and to derive inequalities bounding remaining variables in the resulting single equation. The following example illustrates this technique:

(11) Let $n_1 = Epqr$ and $n_2 = Es$, where p, q, r, s are primes relatively prime to E. Then denoting $\sigma(E)$ by S, we have

$$S(p + 1)(q + 1)(r + 1) = S(s + 1) = E(pqr + s) . \tag{12}$$

Eliminating s and solving for r there result

$$s = (p + 1)(q + 1)(r + 1) - 1 \tag{13}$$

and

$$r = \frac{(S - E)(p + 1)(q + 1) + E}{Epq - (S - E)(p + 1)(q + 1)} . \tag{14}$$

But $S > E$, so from (14)

$$q > \frac{(S - E)(p + 1)}{Ep - (S - E)(p + 1)} . \tag{15}$$

But $q > 0$, so

$$p > (S - E)/(2E - S) , \tag{16}$$

and since $p > 0$

$$S < 2E . \tag{17}$$

From symmetry and since p, q, r are assumed distinct and we are interested ultimately only in prime p, q, r, we take

$$r > q + 1 \tag{18a}$$

and

$$q > p + 1 . \tag{18b}$$

(18a) and (14) give an inequality quadratic in q from which, solving for q,

$$q < \frac{2(S - E)(p + 1)}{Ep - (S - E)(p + 1)} . \tag{19}$$

From (18b) and (19)

$$p < 3(S - E)/(2E - S) . \tag{20}$$

The procedure treats E as an arbitrary input datum subject to (17). In practice it is chosen to obey additional criteria depending on what is being sought. Then (16) and (20) bound p, and (15) and (19) bound q in a (p_i, q_{ij}) domain. The detailed course of the procedure at this point may vary depending on machine and programming limitations. Suffice it to say that as many prime p, q, r as feasible are found using (14)–(20) and a combination of table look-up and sieve routines.

Routines for many specific forms of (4) were programmed and run with varying degrees of success. It should be noted here that amicable pairs are impossible for some forms; consider, for example, the form $n_1 = Epq$, $n_2 = Er^2$. For this form $(p + 1)(q + 1) = r^2 + r + 1$, which is odd for any r. But $(p + 1)(q + 1)$ cannot be odd if p and q are distinct primes.

The 264 new pairs of amicable numbers found are listed in Table I according to Escott's classification [2]. Escott [2], Poulet [3] and García [4] list all known pairs prior to those here given.**

In Table 1 an amicable pair, n_1, n_2 is given simply as n_1, n_2 unless n_1 and n_2 have a greatest common divisor, $u > 1$, in which case the format is u, n_1, n_2. All numbers are in fully decomposed form with E denoting exponentiation, e.g., pair 2 in conventional notation is

$$\begin{Bmatrix} 2^3 \cdot 19 \cdot 83 \cdot 137 \cdot 218651 \\ 2^3 \cdot 19 \cdot 137 \cdot 18366767 \end{Bmatrix} .$$

TABLE 1

TYPE EPQ,ER	2 PAIRS
1 2E8.1039,503.1047311,527845247	2 2E3.19.137,83.218651,18366767
TYPE EPQ,ERS	19 PAIRS
3 2E2.11.23,131.36988691,3041. 1605031	4 2.5E2.23,19.4139,137.599
5 2E2.13.17,233.351287,5507.14923	6 3E2.5E2,7.4049,11.2699
7 2E5.67,547.151992179,24659. 3377603	8 3E2.5.7,71.5879,223.1889
9 2E5.67,719.6608209,2099.2265671	10 3E2.5.7,71.4339,239.1301
11 2E5.79,167.21209129,2659.1339523	12 3.5.7.11,503.1319,769.863
13 2E5.107,83.3333263,1061.263647	14 3.5.7.11,347.23099,449.17863
15 2E11,2351.7095551,16127.1034767	16 3.5.7.11,293.5279,1231.1259
17 3.5.7.11,233.1019479,1091.218459	18 2E9,947.14591,1367.10111
19 3E2.5.13.19,31.184337,263.22343	20 2E5.79,163.1094939,4919.36497
21 3E4.7.11E2.19E2.127,359.144779,911.57149	

** While this paper was in preparation, it was brought to the author's attention that nine new pairs had been found by Alanen, Ore and Stemple (this journal, April 1967).

Table 1—*Continued*

	TYPE EPQR,ES		5 PAIRS
22	2E7,149.1151.3499,604799999	23	2E4,17.137.9319,23150879
24	2E8,257.33151.4259903, 36435879051263	25	2E7,199.359.20411,1469663999
26	2E8,383.809.14083,4380687359		
	TYPE EPQR,EST		68 PAIRS
27	2.5.13,19.83.129011,5039.43003	28	2.5.13,17.179.5381,2339.7451
29	2.5.13,19.89.70979,1039.122849	30	2.5.13,17.197.2339,1619.5147
31	2.5.17,11.89.227629,1699.144611	32	2.5.17,11.101.1889,1427.1619
33	2.5.19,11.41.34154399,9629.1787519	34	2.5.17,13.41.23459,3331.4139
35	2.5.19,11.61.538649,139.2862539	36	2.5.17,13.47.2549,359.4759
37	2.5E2,11.137.38149,19.3158819	38	2.5.19,11.47.27739,359.44383
39	2.5E3,7.4049.143509,359.12915899	40	2.5.19,11.59.15199,151.71999
41	2.7.17,5.47.33195287,1181.8088191	42	2.5E2,7.149.281,179.1879
43	2E2.19.37,41.1109.11369,11.44172449	44	2.5E2,7.149.12671,109.138239
45	2E2.23.53,17.2437.5179,11.18943259	46	2.5E2,7.1499.1667,71.277999
47	2E3.29,17.1217.20939,251.1821779	48	2.5E2,11.23.599,59.2879
49	2E3.29,19.107.233159,647.777199	50	2.5E2,13.17.3499,59.14699
51	2E5,37.2111.109661,223.39290327	52	2.7.17,5.47.173501,1223.40823
53	2E5,47.739.383261,109.123758783	54	2.7.17,5.101.797,113.4283
55	2E5,59.1427.168527,71.200548319	56	2E2.11,17.29.16631,263.34019
57	2E6,67.19963.577151,1151.680133551	58	2E2.53,5.17.62327,251.26711
59	2E6,67.26783.69061,1151.109187021	60	2E3,29.47.9631,13.990719
61	2E6,89.22343.184999,223. 1660837499	62	2E5,37.269.13999,797.133823
63	2E6,113.4567.17351,151.59447951	64	2E5,37.271.1637,1481.11423
65	2E7,271.5939.22536079,251. 144488467199	66	2E5,37.383.599,1709.5119
67	2E5,43.151.104579,479.1457147	68	2E5,43.263.9539,211.522719
69	2E5,43.269.1063,251.50159	70	2E5,47.167.1619,251.51839
71	2E5,53.127.349,479.5039	72	2E5,53.127.76207,197.2660351
73	2E5,59.71.9533,1259.32687	74	2E5,59.71.14783,1187.53759
75	2E5,59.79.56099,439.611999	76	2E5,59.83.571,1637.1759
77	2E5,59.83.1619,449.18143	78	2E5,59.109.1559,199.51479
79	2E5,59.167.857,131.65519	80	2E5,59.317.2591,89.549503
81	2E5,59.1451.57119,71.69115199	82	2E5,59.1481.31799,71.39272999
83	2E5,59.2207.3947,71.7264319	84	2E5,61.89.45343,239.1054247
85	2E5,67.383.11027,71.3999487	86	2E5,71.89.839,191.28349
87	2E5,71.103.14879,127.870479	88	2E5,71.131.1567,107.137983
89	2E5,73.971.17959,59.21530447	90	2E5,97.127.701,83.104831
91	2E6,101.239.35339,599.1441871	92	2E6,101.311.1511,503.95471
93	2E6,101.461.991,373.124991	94	2E6,101.691.2351,251.658783
	TYPE EPQR,ESTU		56 PAIRS
95	2.5E2,17.349.12967,19.29.136163	96	2.5,7.89.359,23.59.179
97	2.5.17,11.101.3659,17.719.6221	98	2.5,7.131.2339,19.53.2287
99	2E2.11,17.107.1038311,37.4751. 11177	100	2.5,7.163.449,19.59.491
101	2E2.11,19.1259.2969,29.149.16631	102	2.5,11.41.239,17.29.223
103	2E2.13,17.1399.51479,19.103.623699	104	2E2,5.17.797201,41.73.27701
105	2E3,11.53.29818669,41.1109.414467	106	2E2,5.41.105071,23.43.25073
107	2E3,11.103.1732799,31.607.11149	108	2E3,11.59.173,53.137.1399

109 2E3,11.139.223439,31.251.46549
110 2E3,11.67.164429,53.101.24359
111 2E3,11.227.114847,31.151.64601
112 2E3,11.71.11689,59.83.2003
113 2E3,13.1061.102499,19.353.215249
114 2E3,11.79.995651,53.83.210719
115 2E3,13.2789.146383,19.293.972407
116 2E3,11.163.191,31.91.11807
117 2E3,13.9371.7391159,29.47.673457861
118 2E3,11.211.503,47.83.317
119 2E3,13.9521.556159,29.47.51486511
120 2E3,11.499.29429,29.149.39239
121 2E3,13.11807.6742199,19.271.204883559
122 2E3,11.1877.2447,31.101.16901
123 2E3,17.43.639007,23.151.138731
124 2E3,13.89.55579,29.97.23819
125 2E4,23.7649.128393,53.449.970087
126 2E3,13.431.1511,23.107.3527
127 2E4,23.13967.1197649,59.239.27881291
128 2E3,13.863.6029,23.89.33767
129 2E4,29.1231.121787,59.83.893111
130 2E3,17.47.2239,23.167.479
131 2E4,29.3583.335213,41.191.4469519
132 2E3,17.53.1039,23.179.233
133 2E3,17.59.5641,19.433.701
134 2E3,17.79.769,29.43.839
135 2E3,17.263.34949,19.59.138401
136 2E3,17.521.24659,19.53.214541
137 2E3,17.601.3659,19.53.36721
138 2E3,17.659.2441,19.53.26861
139 2E3,17.719.1889,19.53.22679
140 2E3,19.71.9719,23.47.12149
141 2E3,19.167.251,23.41.839
142 2E4,29.593.13137,53.109.39383
143 2E4,29.659.11243,59.89.41227
144 2E4,29.661.10799,59.89.39719
145 2E4,37.107.21559,43.227.819
146 2E4,47.107.3767,53.71.5023
147 3E2.5,7.19.2663,11.73.479
148 3E3.5,7.53.467,17.47.233
149 3E3.5,7.89.3347,17.29.4463
150 3E3.5,11.17.227,23.37.53

TYPE EPQRS,ET 7 PAIRS

151 2E2.11,13.47.6829.421079,1932656140799
152 3.5.7,11.13.37.3779,24131519
153 2E4,17.163.1013.3607,10799927423
154 2E4,17.137.9649.269131,6451255519199
155 2E4,17.137.14843.25013,922328614943
156 3E2.5.13,11.23.79.1051,24238079
157 3E2.5.13,11.19.211.14699,747935999

TYPE EPQRS,ETU 19 PAIRS

158 2E2.11,13.53.8329.84061,461.1145841479
159 2.5,11.17.19.47,239.863
160 2E2.13,17.23.467.36529,53.136768319
161 2E2.19,13.17.37.109,263.3989
162 2E3,13.17.449.3387971,28619.13424039
163 2E3,11.41.191.271,83.313343
164 2E3,13.17.479.92177,6803.1638719
165 2E3,13.19.61.4517,3347.28111
166 2E3,13.19.131.69191,2239.1141667
167 2E3,13.19.83.347,587.16703
168 2E3,13.19.1993.26099,149.97147679
169 2E3,17.19.41.2027,3041.10079
170 2E3,13.23.59.4594127,20327.4556159
171 2E3,17.23.37.173,1367.2087
172 2E3,17.29.37.18311,71.5218919
173 2E3,19.23.29.223,1439.2239
174 2E3,19.23.29.47189,197.3431999
175 2E3,19.23.53.191,71.69119
176 2E3,19.23.223.8861,31.29776319

TYPE EPQRS,ETUV 23 PAIRS

177 2.5,11.23.79.7109,17.79.113759
178 2.5,11.19.89.383,17.359.1279
179 2.5,13.23.139.63737,11.5879.42491
180 2.5E2,13.41.10799,23.251.1049
181 2.5,17.19.71.90149,11.647.300499
182 2.5E2,13.73.10799,19.167.3329
183 2.5E2,13.59.725999,19.307.98999
184 2.5E2,13.79.1619,19.251.359

TABLE 1—*Continued*

185 2.7,5.11.97.26212247,23.6803.1132627
186 2.7,11.13.29.47,19.23.503
187 2.7.11,13.191.5939,19.307.2591
188 2.7.11,17.149.3079,19.53.7699
189 2E2,5.13.167.179657,53.1511.31051
190 2.11,5.23.43.67,7.197.271
191 2E2.17,23.41.1289.9043,13.89.9333407
192 2E2,5.23.263.587,41.47.11087
193 2E2,5.17.461.409867,37.107.4983131
194 2E2,5.23.83.227,47.71.797
195 2E2.19,17.97.773.7789,11.113.7774829
196 2E3,17.19.7699.1261459,47.769.94609499
197 2E2,5.17.467.36061,37.107.444131
198 2E2,5.23.31.123191,53.191.54751
199 2E2.19,23.37.569.121469,11.151.34618949

TYPE EPQRS,ETUVW 2 PAIRS

200 2.17,5.23.1223.72901,7.11.67.1968353
201 2E2,11.23.521.10289,13.17.239.27577

TYPE EPQRST,EUVWX 1 PAIR

202 2E2,19.29.41.107.1667,13.17.59.300239

TYPE MISCELLANEOUS 62 PAIRS

203 2E5.61.15472687,2E9.2351.25117
204 2E5.47.76961,2E6.431.4241
205 2E8.239.1019.1373,2E7.674029439
206 2E5.199.464819,2E6.53.853999
207 2E3.13.1913.5417,2E5.179.192037
208 2E5.71.3245579,2E8.269.106703
209 2E3.23.157.3477869,2E5.14051.130349
210 2E5.61.4861999,2E9.4079.4549
211 2E3.17.8161.14621,2E5.43.11624489
212 2E5.619.967,2E9.83.439
213 2E3.23.173.29021,2E5.859.189407
214 2E3.29.109.433,2E4.19.34649
215 2E4.29.83.63377,2E5.557.140839
216 2E3.13.521.569,2E5.449.2203
217 2E4.29.9091.69341,2E6.179.25648531
218 2E3.13.653.2621,2E5.229.24851
219 2E4.43.113.51461,2E9.1367.5717
220 2E3.17.137.293,2E5.83.2069
221 2E5.17.1949.4493,2E3.41.15773939
222 2E4.13.41.2609,2E3.587.5393
223 2E5.67.1489.2014379,2E7.239.210099833
224 2E4.29.257.2539,2E6.773.6199
225 2E5.83.2729.4283,2E11.293.51407
226 2E5.13.193.709,2E3.283.28517
227 2E3.17.4157.16433,2E5.53.241.22409
228 2E5.61.163.2539,2E6.1301.9839
229 2E3.17.7589.7457561,2E5.53.229.19531709
230 2E3.29.61.569,2E4.37.89.149
231 2E4.53.109.14699,2E5.59.461.1549
232 2E3.53.61.433,2E4.29.41.557
233 2E4.61.139.440529893,2E5.41.25171.1779709
234 2E3,11E2.67.71.113.5711
235 2E2,13E2.5.743.23089571,53.35023.9973133
236 2,5E2.59.503,5.11.13.929
237 2E3,13E2.383.2309,23.239.28181
238 2,5E2.7.59.71,5.47.3719
239 2E2,29E2.14207.288413,5.17.311.105922711
240 2,5E2.13.89.167,5.11.91139
241 2E3,19E2.9338111,13.863.294131
242 2,7E2.13.31.313,7.5.167047
243 2E3.37,17E2.59.315461,5810810039
244 2,7E2.13.107.967,7.5.37.45737
245 2,5E2.13.1542239,5.11.59.154937
246 2.11,5E2.347.5939,5.929.11483
247 2,5E2.23.397.233279,5.7.11159.128951
248 2.13,5E2.37.21059,5.269.15313

249 2E3,23E2.31.5381.10429,23.97.422343479
250 2.5,7E3.4211,19.179.467
251 2.13,5E2.89.116531,5.53.1003469
252 2.7,5E3.727271,11.251.37517
253 3E2.5,7E2.4643.4658359,7.773.199144889
254 3E2,5E3.31.4259,5.7.11.36919
255 3E2.5,7E2.1499.29567,7.1231.256499
256 2,5E3.11.1549,5E2.19.4679
257 2,5E3.19.1640519,5E2.11.2099.6551
258 5,3E4.7.19.32939,3E9.29.719
259 2,5E3.3719.4679,5E2.7.107.101399
260 37,2E5.73.449,2E3.89.1553
261 5,3E4.7.2143.319499,3E3.19.23.34535819
262 7.13,3E2.11.631,3.5E3.157
263 2,5E3.5639.8963959,5E2.7.103.305786699
264 2,7E2.13.1487.32117,7.5E2.211.725381

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