The author is indebted to C. F. J. Outred for, among other things, the notion of rotation. The referee has pointed out that in the table for P_n ($= G(n)/n$ in the present notation) of [1, p. 397] the last entry should read 12198 instead of 12196. There are further references in [1].

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The Maxima of $P_r(n_1, n_2)$

By M. S. Cheema* and H. Gupta

1. In this note, we study the maxima of $P_r(n_1, n_2)$, the number of partitions of the vector (n_1, n_2) into exactly r parts (vectors) with positive integral components.

The generating function $\phi_r(x_1, x_2)$ for $P_r(n_1, n_2)$ is given by

$$\prod_{k_1,k_2=1}^{\infty} (1 - zx_1^{k_1}x_2^{k_2})^{-1} = 1 + \sum_{r=1}^{\infty} z^r \phi_r(x_1, x_2) \tag{1.1}$$

$$\phi_r(x_1, x_2) = 1 + \sum_{n_1,n_2=1}^{\infty} P_r(n_1, n_2)x_1^{n_1}x_2^{n_2} . \tag{1.2}$$

2. If $q_r(n_1, n_2)$ denotes the number of partitions of (n_1, n_2) into at most r parts (vectors) with nonnegative integral components, then it follows that $q_r(n_1, n_2) = P_r(n_1 + r, n_2 + r)$. It is clear that $q_r(n_1, n_2)$ is an increasing function of r for $1 \leqq r < n_1 + n_2$, and becomes constant for $r \geqq n_1 + n_2$, on the other hand $P_1(n_1, n_2) = 1$ and $P_r(n_1, n_2) = 0$ for $r > \min(n_1, n_2)$. From the table of values of $P_r(n_1, n_2)$ computed by Cheema, we notice that for $n_1 \geqq n_2 > 0$, there is a unique s such that

$$P_1(n_1, n_2) < P_2(n_1, n_2) < \cdots < P_s(n_1, n_2) \geqq P_{s+1}(n_1, n_2) \geqq \cdots \geqq P_{n_2}(n_1, n_2) .$$

We use s in this sense in all that follows. The values of s were computed for all n_1, $n_2 \leqq 50$. We might remark that a similar conjecture holds for the number of partitions of n into exactly r summands. An explicit formula for $P_r(n_1, n_2)$ for general r is not known, $P_r(n_1, n_2)$ do satisfy a recurrence relation and behave very much like a polynomial in n_1, n_2, i.e., $P_r(n_1, n_2)$ is a semipolynomial of degree $r - 1$ in n_1 and n_2 relative to modulus $r!$ as shown by Wright [2]. Thus

$$P_r(n_1, n_2) = \sum_{t_1=1}^{r} \sum_{t_2=1}^{r} \beta(t_1, t_2, n_1, n_2)n_1^{t_1-1}n_2^{t_2-1} ,$$

where $\beta(t_1, t_2, n_1, n_2)$ depends on r, t_1, t_2 and on the residues of n_1, n_2 to moduli 1, 2, 3, $\cdots$, $[r/t_i]$, but not otherwise on n_1, n_2. A rough estimate for s is obtained by studying the maxima of a function which behaves very much like $P_r(n_1, n_2)$.

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3. For n_1, n_2 large compared to r, $P_r(n_1, n_2)$ behaves very much like the function

$$\frac{1}{r!}\binom{n_1-1}{r-1}\binom{n_2-1}{r-1}.$$

Using this estimate and using $P_r(n_1, n_2) \geqq P_{r+1}(n_1, n_2)$, we obtain $s = \min(r, n_1, n_2)$, where r is the least positive integer satisfying

$$(3.1) \qquad (n_1 - r)(n_2 - r) \leqq r^2(r+1).$$

Roughly such an r is given by $(n_1 n_2)^{1/3}$. If $n_1 = n_2 = n$, then as in [1]

$$(3.2) \qquad P_r(n, n) \simeq \frac{1}{r!}\binom{n-1}{r-1}^2 \exp\left(\frac{r^3 \log r}{n^2}\right).$$

Hence $P_s(n, n) \geqq P_{s+1}(n, n)$ implies that

$$(3.3) \qquad (n-s)^2 \leqq (s+1)s^{2-(3s^2+3s+1)/n^2}.$$

As a rough estimate we have $s \simeq n^{2/3}$. The inequality (3.3) gives a good estimate for s for a particular n. Thus for $n = 50$, the value of s by (3.3) is 14, while the actual value is 13. For $n = 52$, $s = 14$ both by the inequality and the tables.

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