Very similar methods can be applied for the generation of the modified Bessel functions $I_n(x)$ and $K_n(x)$.

With the improvement just described the widely used recurrence techniques are very straightforward methods for the generation of the sets of Bessel functions with real argument x and varying index n.

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1. I. A. STEGUN & M. ABRAMOWITZ, "Generation of Bessel functions on high speed computers," *MTAC*, v. 11, 1957, pp. 255–257. MR **20** #459.
2. M. GOLDSTEIN & R. M. THALER, "Recurrence techniques for the calculation of Bessel functions," *MTAC*, v. 13, 1959, pp. 102–108. MR **21** #4530.
3. M. ABRAMOWITZ & I. A. STEGUN, (EDITORS), *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1966, pp. 386, 452.
4. C. W. CLENSHAW, *Chebyshev Series for Mathematical Functions*, Nat. Phys. Lab. Math. Tables, Vol. 5, Dept. Scientific and Indust. Research, Her Majesty's Stationery Office, London, 1962. MR **26** #362.
5. C. HASTINGS, JR. & J. P. WONG, JR., "Analytical approximations," *MTAC*, v. 8, 1954, p. 46.
6. A. J. M. HITCHCOCK, "Polynomial approximations to Bessel functions of order zero and one and to related functions," *MTAC*, v. 11, 1957, pp. 186–188. MR **19**, 64.
7. J. WIMP, "Polynomial expansions of Bessel functions and some associated functions," *Math. Comp.*, v. 16, 1962, pp. 446–458. MR **26** #6452.

The Zeros of P_ν^1 (cos θ) and $\frac{\partial}{\partial\theta} P_\mu^1$ (cos θ)*

By Peter H. Wilcox

Introduction. In the course of a recent study [1] of the scattering of an electromagnetic wave by a semi-infinite, perfectly conducting cone, it became necessary to compute numerically sets of positive zeros of certain associated Legendre functions treated as functions of their degree; that is, to find ν_i and μ_i, $i = 1, 2, 3, \cdots$, satisfying

$$P_{\nu_i}^1(\cos\theta) = 0\,, \tag{1}$$

and

$$(\partial/\partial\theta)\, P_{\mu_i}^1(\cos\theta) = 0\,, \tag{2}$$

for a given θ. The method presented here employs a trigonometric series expansion for the Legendre functions to obtain these zeros.

Formulas. An expression for the associated Legendre function valid for $0 < \theta < 180°$ is [2]

$$\begin{aligned} P_\nu^\mu(\cos\theta) &= \pi^{-1/2}2^{\mu+1}(\sin\theta)^\mu \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)} \\ &\quad\cdot \sum_{k=0}^{\infty} \left\{ \frac{(\mu+1/2)_k(\nu+\mu+1)_k}{k!(\nu+3/2)_k} \sin\left[(\nu+\mu+2k+1)\theta\right] \right\}. \end{aligned} \tag{3}$$

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TABLE I

The first 50 zeros of the Legendre function and its derivative

i	$\nu_i;\ P^1_{\nu_i}(\cos 165^\circ) = 0$	$\mu_i;\ \frac{\partial}{\partial\theta} P^1_{\mu_i}(\cos 165^\circ) = 0$
1	1.0316313	.9671403
2	2.0844338	1.9189013
3	3.1499290	2.8870839
4	4.2230957	3.8878600
5	5.3010868	4.9171089
6	6.3822487	5.9656383
7	7.4655810	7.0264388
8	8.550453	8.095136
9	9.636450	9.169086
10	10.723293	10.246663
11	11.810784	11.326832
12	12.898783	12.408911
13	13.987187	13.492435
14	15.075917	14.577076
15	16.164916	15.662599
16	17.254136	16.748830
17	18.343542	17.835637
18	19.433106	18.922921
19	20.522803	20.010602
20	21.612615	21.098619
21	22.702527	22.186921
22	23.792526	23.275468
23	24.882601	24.364227
24	25.972743	25.453171
25	27.062943	26.542276
26	28.153197	27.631524
27	29.243498	28.720898
28	30.333840	29.810384
29	31.424221	30.899971
30	32.514636	31.989647
31	33.605082	33.079405
32	34.695557	34.169236
33	35.786057	35.259134
34	36.876580	36.349093
35	37.967126	37.439106
36	39.057691	38.529170
37	40.14827	39.61928
38	41.23887	40.70943
39	42.32949	41.79963
40	43.42012	42.88986
41	44.51076	43.98012
42	45.60142	45.07041
43	46.69209	46.16073
44	47.78277	47.25108
45	48.87345	48.34146
46	49.96415	49.43186
47	51.05486	50.52228
48	52.14557	51.61272
49	53.23630	52.70318
50	54.32703	53.79366

Direct substitution of $\mu = 1$ does not lead to useful results, since the series then diverges. However, if $S_\nu(\theta)$ is defined as

$$S_\nu(\theta) = \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\nu)_k}{k!(\nu + 3/2)_k} \sin[(\nu + 2k)\theta], \tag{4}$$

then letting $\mu = -1$ in Eq. (3) gives

$$P_\nu^{-1}(\cos\theta) = \frac{1}{\sqrt{\pi}\sin\theta} \frac{\Gamma(\nu)}{\Gamma(\nu + 3/2)} S_\nu(\theta). \tag{5}$$

Since [3]

$$P_\nu^{-m}(z) = \frac{\Gamma(\nu - m + 1)}{\Gamma(\nu + m + 1)} (-1)^m P_\nu^m(z) \tag{6}$$

(for integer m), Eq. (3) finally becomes

$$P_\nu^1(\cos\theta) = \frac{-1}{\sqrt{\pi}\sin\theta} \frac{\Gamma(\nu + 2)}{\Gamma(\nu + 3/2)} S_\nu(\theta). \tag{7}$$

An expression for the derivative of the Legendre function found from Eq. (7) and from the recurrence relation

$$\frac{\partial}{\partial\theta} P_\nu^1(\cos\theta) = \frac{1}{\sin\theta} [\nu\cos\theta P_\nu^1(\cos\theta) - (\nu + 1)P_{\nu-1}^1(\cos\theta)] \tag{8}$$

may be written in the form

$$\frac{\partial}{\partial\theta} P_\nu^1(\cos\theta) = \frac{-\nu}{\sqrt{\pi}\sin^2\theta} \left[\frac{\Gamma(\nu + 2)}{\Gamma(\nu + 3/2)}\right] \Big/ \left[\cos\theta S_\nu(\theta) - \left(1 + \frac{1}{2\nu}\right) S_{\nu-1}(\theta)\right]. \tag{9}$$

The expressions in (7) and (9) may be evaluated on a digital computer, and so the zeros of the respective functions may be found by an iterative technique (e.g., that of Newton-Raphson). The series involved in either case does converge, although slowly, and accurate results can be obtained if enough terms are included. Since the parameter, θ, enters only in the argument for the sine function, it does not affect the rate of convergence. Methods exist for evaluating the gamma functions, although they are not needed if only the zeros are required.

Results. The zeros were calculated at the University of Michigan Computing Center on an IBM 7090 computer in double precision. This would indicate a possible accuracy of 14 to 16 significant figures. Actually, the magnitude of the terms in $S_\nu(\theta)$ decreases so slowly that results of this accuracy would require so many terms as to be impractical when many zeros are required. But if 500 terms are used in the series, the coefficient of the sine function in the last term varies from less than 10^{-8} when ν is very small up to about 2×10^{-6} when $\nu = 100$. This leads to 7 or 8 significant figures in the zeros themselves over the whole range. In general, the time required to compute $S_\nu(\theta)$ with 500 terms is about 0.4 to 0.6 seconds, although a considerable savings in time may be had in those instances where $\sin(2k + \nu)\theta$, $k = 0, 1, 2, \cdots$, is a repeating function in k such that a small table of values may be used for it.

When the zeros for one value of θ are found in ascending order, the separation between successive zeros becomes nearly constant. In fact, it can be shown that [4]

$$(10) \qquad |\xi_{i+1} - \xi_i - (\pi/\theta)| < \epsilon$$

for any arbitrarily small ϵ if i is large enough, where ξ_i represents either ν_i or μ_i. Thus, when two successive zeros have been found, the next can be easily estimated. For higher-order zeros this estimate is usually good enough so that only three function-evaluations are needed for the correction term in the iteration to be less than 10^{-8}. Also, when only one zero is known, π/θ can be used as an increment to get an estimate of an adjacent zero.

Since Eq. (3) holds for $0 < \theta < 180°$, it should be possible to find zeros for any angle in this range. In connection with our work, we have found the zeros, ν_i and μ_i, for $i = 1, 2, 3, \cdots, 100$ at a number of angles between 150° and 172.5° and also some low-order zeros at angles from 2.5° to 177.5°. Table I shows the first 50 zeros at 165°, which were computed by carrying $S_\nu(\theta)$ to 500 terms or until the magnitude of the coefficient became less than 10^{-8}. As a test, the same zeros were found using 2000 terms in the series, which in general, represents a decrease of two orders of magnitude in the size of the last term. As a second test, the expressions in (7) and (9) were evaluated for integer values of ν, and the results compared with values calculated using the recurrence relations for the Legendre functions. From these tests, the number of significant figures present in the values for the Legendre functions and in the zeros themselves was determined. The tests verified the accuracy of the numbers in Table I.

Of the various tables and bounds that have been published relating to these zeros [4]–[7], only Waterman's work [7] provides results of precision comparable with those given here. His values for ν_i $(i = 1, 2, \cdots, 30)$ at 165° agree with ours to six and usually seven significant figures, the latter being the limit of his tables. However, for μ_i $(i = 2, 3, \cdots, 30)$, his zeros are consistently lower, with the difference showing up in the fifth decimal place for the higher-order zeros. In each case where such a discrepancy exists, the representations in (7) and (9) have been used to evaluate the appropriate Legendre function at each of the two proposed values for the zero. In all instances, the resulting values were significantly smaller for the zeros listed in Table I than for the values presented by Waterman. From this, it is concluded that the zeros given here are more accurate.

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1. T. B. A. Senior & P. H. Wilcox, "Traveling waves in relation to the surface fields on a semi-infinite cone," *Radio Sci.*, v. 2, 1967, pp. 479–487.

2. M. Abramowitz & I. A. Stegun, (Editors), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Math. Series, No. 55, U. S. Government Printing Office, Washington, D. C., 1964, p. 335. MR **29** #4914.

3. W. Magnus & F. Oberhettinger, *Formulas and Theorems for the Special Functions of Mathematical Physics*, translated from German, Chelsea, New York, 1949. MR **10**, 532.

4. P. A. Carrus & C. G. Treuenfels, "Tables of roots and incomplete integrals of associated Legendre functions of fractional orders," *J. Math. Physics*, v. 29, 1951, pp. 282–299. MR **12**, 638.

5. K. M. Siegel, J. W. Crispin, R. E. Kleinman & H. E. Hunter, "The zeros of $P_{n_i}'(x_0)$ of non-integral degree," *J. Math. Physics*, v. 31, 1952, pp. 170–179. MR **14**, 321.

6. K. M. Siegel, J. W. Crispin, R. E. Kleinman & H. E. Hunter, "Note on the zeros of $(dP_{n_i}^1(x)/dx)|_{x=x_0}$," *J. Math. Physics*, v. 32, 1953, pp. 193–196. MR **15**, 255.

7. P. C. Waterman, "Roots of Legendre functions of variable index," *J. Math. Physics*, v. 42, 1963, pp. 323–328. MR **28** #2269.