used in the approximations which are closely connected with the problem being studied and which can also be used in accelerating the convergence of iterative methods of Picard-Neumann-Banach type.

The book consists of seven chapters in which the theory and the application of the method of moments is investigated. In Chapter I, "Approximation of bounded linear operators," the author introduces the concepts of an abstract Hilbert space, bounded linear operators, discusses without proofs their properties, formulates the method of moments in a Hilbert space and shows its relation to the projection method or the abstract Ritz-Galerkin method. In Chapter II, "Equations with completely continuous operators," the method of moments is first formulated for completely continuous operators and then it is applied to the solution of nonhomogeneous equations with completely continuous linear operators and to the determination of their eigenvalues. It is also shown how the method of moments can be used in the acceleration of convergence of iterative methods of Picard-Neumann-Banach type. In Chapter III, "The method of moments for self-adjoint operators," the problem of moments is first formulated for equations involving self-adjoint operators and then the method is applied to the determination of the spectrum of a self-adjoint operator and to the solution of nonhomogeneous linear equations involving bounded self-adjoint operators. In Chapter IV, "Speeding up the convergence of linear iterative processes," the author first discusses the linear iterative processes, $x_{n+1} = Ax_n + f$, and various methods for their acceleration in the solution of the equation $x = Ax + f$ and then shows how the method of moments may be used to speed up the convergence of the above linear iterative processes. He applies this technique to the solution of the finite-difference equations arising in the numerical solution of the first boundary-value problem for an elliptic operator with constant coefficients. In Chapter V, "Solution of time-dependent problems by the method of moments," the author applies the method of moments to the solution of various classes of nonstationary linear problems (e.g., oscillatory systems with a finite number of degrees of freedom, heat conduction in an inhomogeneous rod, the transient in an automatic control system, etc.). In Chapter VI, "Generalization of the method of moments," it is shown that the author's method is applicable to certain classes of linear equations involving unbounded operators. In Chapter VII, "Solution of integral and differential equations," the method of moments is applied to the solution of certain classes of linear integral equations in L_2 spaces and boundary-value problems for ordinary and partial differential equations. The numerical results obtained by the method of moments are illustrated by actually solving approximately the problems associated with bending of a beam of variable cross-section and with the field of an electrostatic electron lens.

Finally, it should be noted that the monograph is clearly written and well motivated. Its English translation is quite satisfactory.

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8 [2.55].—RAMON E. MOORE, *Interval Analysis*, Prentice-Hall, Englewood Cliffs, N. J., 1966, xi + 145 pp. Price $9.00.

In the days when numerical computations were made by hand on desk computers, it was generally customary to monitor the degree of precision of all intermediate and final results by a simple set of rough rules; the precision of the result of each step was estimated in terms of the precision of the operands, and the last significant digit was indicated by an underline when this result was written down. Proceeding in this way, one obtained an estimate of the significance of the final results. The procedure failed to take into account the accumulation of rounding errors, but it provided a check on possible loss of significance by subtraction, which is an ever-present danger in a computation of any size, and it certainly was enormously better than the complete absence of any significance-monitoring procedure, which is almost universal practice with today's multi-million-dollar automatic digital computers.

For the enormous numerical calculations that are commonly attempted today, quite sophisticated significance-monitoring procedures will clearly be needed before one can take such calculations seriously. Before one can develop generally satisfactory procedures of this kind, an extensive study of the overall problem of significance-monitoring in large calculations is necessary. Professor Moore's development of methods using interval-arithmetic over the past seven to eight years represents substantial progress in this area. The book under review presents a detailed theory and analysis of these methods and is undoubtedly the most important single work to appear in the field.

As Moore says, practically all ways of expressing the accuracy of a quantity involve an interval $[a, b]$ (e.g., $[p - e, p + e]$ where p equals expected value, e equals probable error, or the like) together with the statement that the true value lies in that interval or lies in it with a stated probability. Roughly speaking, there are three classes of methods for assigning an interval to the result of an arithmetic operation, when the intervals assigned to the operands are given. At one extreme, there is "significance arithmetic" (see Ashenhurst and Metropolis, "Unnormalized floating-point arithmetic," *J. Assoc. Comput. Mach.*, v. 6, 1959, p. 415), in which the accumulation of errors is ignored, and it is simply assumed that the error of a result is influenced by one operand, but not by both. (For example, in multiplication, the number of significant digits of the product is taken as the number of significant digits of the less accurate factor.) Intermediate is the procedure, attributed to L. H. Thomas, in which a probable error is carried along with each quantity, and when two quantities are combined the error of the result is calculated on the assumption that the errors of the operands are normally distributed and uncorrelated. (Here, if n nearly identical quantities are added, the estimated error of the result increases as $\sqrt{n}$.) At the second extreme is Moore's "interval arithmetic," in which the interval assigned to the result is *certain* to contain the true value of the result, if the same was true of the operands. For example, if two intervals $[a, b]$ and $[c, d]$ are added, the resulting interval is $[a + c, b + d]$. When the calculation is done on a finite computer, the upper limit of the interval is then rounded up (when necessary) and the lower limit is rounded down (when necessary). In this way, guaranteed error bounds for the result of a computation are obtained, if guaranteed error bounds were available for the input data. The procedure is fully automatic when suitably programmed, two numbers a and b being stored for each quantity, corresponding to the interval $[a, b]$ assigned to the quantity. (Here, if n nearly identical quantities are added, the estimated error of the result increases as n.) In any of these systems, the errors of finite-difference methods or other approximate numerical methods must be analyzed independently.

The chief difficulties that have arisen in connection with the first two classes of methods mentioned above appear to arise from the fact that even if one stores in the computer, in some fashion, an estimate of the probable error of each quantity, this gives no information about the correlations of the errors of pairs of quantities; these correlation effects can produce cumulative results in some computational algorithms. For this reason, the study of interval analysis (which in effect allows for the worst possible degree of correlation at each step) seems especially important.

One of the main problems attacked is this: For the calculation of a quantity or set of quantities (inverse of a matrix, value of an algebraic function, root of an algebraic equation, solution of a differential equation, coefficients of a power series, or the like), there are generally many algorithms that are equally satisfactory from the point of view of the required amount of calculation and the accuracy that would be attained if the input data were infinitely precise and all arithmetic steps were performed with infinite accuracy (infinitely many binary or decimal places); the problem is to find, among these, the algorithms that give the narrowest intervals for the final results in a calculation by interval arithmetic. [This is just a precisely defined form of the central problem in any theory of analysis of numerical errors.] Often the optimal algorithm is not a previously known one, and considerable ingenuity is required to construct such algorithms. This activity, of course, overlaps to some extent similar activity on the part of many numerical analysts (in the old-fashioned sense) over the years, but it seems to the reviewer that Moore's approach has several valuable new ingredients: (1) its complete rigor, (2) the power of various theoretical concepts and tools introduced by Moore (e.g., a metric topology for intervals, and interval integrals), and (3) the ability to make precise evaluation and comparison of complicated algorithms by computations on computers programmed for interval arithmetic. Furthermore, errors introduced by mathematical approximation can often be expressed in terms of intervals and thereby incorporated directly and automatically in a calculation; for example, the remainder after n terms of a Taylor's series involves the $(n + 1)$th derivative at an argument x^* for which a precise interval $[a, x]$ is known.

Although interval analysis is in a sense just a new language for inequalities, it is a very powerful language and is one that has direct applicability to the important problem of significance in large computations.

The range of subjects treated is indicated by the chapter headings, which are: Introduction, Interval numbers, Interval arithmetic, A metric topology for intervals, Matrix computation with intervals, Values and ranges of values of real functions, Interval contractions and root finding, Interval integrals, Integral equations, The initial-value problem in ordinary differential equations, The machine generation of Taylor coefficients, Numerical results with the Kth order methods, Coordinate transformations for the initial-value problem. There are two appendices.

The reviewer looked in vain for a comparison of interval arithmetic with other modes of significance arithmetic. It is difficult to resist the conjecture that in some calculations, where errors are mostly uncorrelated, interval arithmetic must give overly pessimistic estimates of the final errors; it would be of value to know whether this can often be the case (it may be quite rare), and if so whether anything can be done about it (for example, possibly a temporary switch to one of the other modes of significance arithmetic).

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9 [2.55, 4, 5].—I. BABUSKA, M. PRAGER, I. VITASEK, *Numerical Processes in Differential Equations*, John Wiley & Sons, Inc., New York, 1966, x + 351 pp., 24 cm. Price $9.50.

This translation from the 1964 Czech edition reads quite well and has the following chapters:

1. Introduction, 4 pages
2. Stability of Numerical Processes and Some Processes of Optimization of Computations, 44 pages
3. Initial-Value Problems for Ordinary Differential Equations, 56 pages
4. Boundary-Value Problems for Ordinary Differential Equations, 150 pages
5. Boundary-Value Problems for Partial Differential Equations of the Elliptic Type, 50 pages
6. Partial Differential Equations of the Parabolic Type, 35 pages.

The stability chapter contains some nice examples of the loss in accuracy due to finite word length. Definitions of stability are given and applied. They boil down to continuous dependence on the inhomogeneous data or specified (polynomial) growth of errors.

Much of the standard convergence and stability (in the sense of Dahlquist) theory for linear multistep and one-step methods is presented in a neat form. Unfortunately, predictor-corrector methods are never mentioned.

Only linear two-point boundary-value problems are considered for second- and fourth-order equations. "Factorization" methods in which the boundary-value problem is replaced by several initial-value problems for first-order equations (in both directions) are studied in some detail. As is later shown, these methods are suggested by the factorization of the tri- and qui-diagonal matrices so familiar in difference methods for such problems. The stability of the initial-value problems is shown under appropriate conditions. The "shooting" method is but briefly mentioned and the usual warning of possible instability is based on an example. A detailed treatment of finite-difference methods, including higher-order accurate schemes, is given. Finally, the Ritz method for self-adjoint positive-definite problems is considered in some generality.

The material on elliptic problems is devoted mainly to setting up difference equations for second-order self-adjoint problems with no mixed derivatives and to solving the difference equations by some of the standard iterative methods (not including alternating directions).

Linear parabolic problems in one- and two-space dimensions are treated briefly using maximum estimates for implicit and explicit schemes and Lee's energy estimates for the Crank-Nicolson scheme in one dimension. Alternating directions are described.