Extensions of Symmetric Integration Formulas*

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1. Introduction. Assume we are given an integration formula for the m-dimensional cube C_m of the form

$$(1) \qquad \int_{-1}^{1} \cdots \int_{-1}^{1} f(x_1, \cdots, x_m) dx_1 \cdots dx_m \simeq \sum_{j=1}^{N} A_j f(\nu_{j1}, \cdots, \nu_{jm})$$

which is exact for all polynomials of degree $\leqq d$; this is equivalent to assuming (1) is exact for all monomials

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}, \qquad \alpha_1, \cdots, \alpha_m \text{ nonnegative integers},$$
$$0 \leqq \alpha_1 + \alpha_2 + \cdots + \alpha_m \leqq d .$$

We say that such a formula (1) has *degree d*.

We say that formula (1) is *symmetric* if the right side of (1) is not changed under any of the $m!$ permutations of the variables $x_1, x_2, \cdots, x_m$. In other words (1) is symmetric provided that if the formula contains the point

$$(\nu_{j1}, \nu_{j2}, \cdots, \nu_{jm}) \qquad \text{coeff. } A_j$$

then the formula also contains the point

$$(\nu_{jp_1}, \nu_{jp_2}, \cdots, \nu_{jp_m}) \qquad \text{coeff. } A_j$$

where $(p_1, p_2, \cdots, p_m)$ is any permutation of $(1, 2, \cdots, m)$.

In this article we show how a symmetric formula (1) of degree $d \leqq 2m + 1$ for C_m can be used to construct a symmetric formula of the same degree for C_n, $n > m$.

Following Hammer and Stroud [2] we say that formula (1) is *fully-symmetric* if the right side of (1) is not changed under any of the $2^m(m!)$ linear transformations of C_m onto itself. Lyness [3] has given a method by which a fully-symmetric formula of degree $d \leqq 2m + 1$ can be used to construct a fully-symmetric formula of degree d for C_n, $n > m$. Lyness defines a formula for C_n constructed by this method as an *extension* of the formula for C_m. The result of this article is a variation of the result of Lyness.

In what follows we use the notation

$$I_{C_m}(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}) \equiv \int_{-1}^{1} \cdots \int_{-1}^{1} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m} dx_1 \cdots dx_m$$

and $V_m \equiv I_{C_m}(1) = 2^m$. We note for future reference that if

$$I_{C_m}(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}) = c_{\alpha_1 \cdots \alpha_m} V_m$$

then

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$$I_{C_n}(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_m^{\alpha_m}) = c_{\alpha_1\cdots\alpha_m}V_n .$$

2. The Method of Extension. We assume formula (1) is symmetric and we write the points and coefficients in this formula as follows:

$$(2) \qquad (\underbrace{\mu_{i1}, \cdots, \mu_{i1}}_{m_{i1}\text{ times}}, \cdots, \underbrace{\mu_{ik}, \cdots, \mu_{ik}}_{m_{ik}\text{ times}}, \underbrace{0, \cdots, 0}_{m - m_i\text{ times}})_S \qquad A_i = a_iV_m ,$$

$m_i = m_{i1} + \cdots + m_{ik}$, $0 \leqq m_i \leqq m$, $1 \leqq m_{i1} \leqq m$, $\cdots$, $1 \leqq m_{ik} \leqq m$, $\mu_{ir} \neq \mu_{is}$ $r \neq s$, $i = 1, 2, \cdots, M$.

Here $(\nu_{i1}, \cdots, \nu_{im})_S$ denotes the set of points consisting of the point $(\nu_{i1}, \cdots, \nu_{im})$ and all points obtained by permuting the coordinates $\nu_{i1}, \cdots, \nu_{im}$ in all possible ways. A_i is the coefficient of each point in the set of points (2).

We define the extension of formula (2) to be the formula for C_n consisting of the following:

$$(3) \qquad (\underbrace{\mu_{i1}, \cdots, \mu_{i1}}_{j_{i1}\text{ times}}, \cdots, \underbrace{\mu_{ik}, \cdots, \mu_{ik}}_{j_{ik}\text{ times}}, \underbrace{0, \cdots, 0}_{n - j_i\text{ times}})_S \qquad B_{i,j_{i1},\cdots,j_{ik}} ,$$
$$j_i = j_{i1} + \cdots + j_{ik} ,$$

for all possible choices of $j_{i1}, \cdots, j_{ik}$ which satisfy $0 \leqq j_{i1} \leqq m_{i1}, \cdots, 0 \leqq j_{ik} \leqq m_{ik}$ and for all i, $i = 1, 2, \cdots, M$. The coefficient of the points (3) is

$$(4) \qquad B_{i,j_{i1},\cdots,j_{ik}} = \frac{(-1)^{m_i - j_i}Z(n, m, m_i, j_i)}{(m_{i1} - j_{i1})!\cdots(m_{ik} - j_{ik})!}\, a_iV_n$$

where $Z(n, m, m_i, j_i) = (n - m + m_i - j_i - 1)!/(n - m - 1)!$.

We now state:

THEOREM 1. *If formula* (2) *for* C_m *has degree* d, *where* $d \leqq 2m + 1$, *then the points* (3) *with coefficients* (4) *are a formula of degree* d *for* C_n, $n > m$.

We do not know how to prove this theorem for all m but we believe it to be true. We have verified it for $m \leqq 5$; we will show how it can be verified for $m = 4$.

To start let us assume that the points (2) have the special form

$$(5) \qquad (\mu_{i1}, \mu_{i1}, \mu_{i2}, \mu_{i2})_S \qquad A_i = a_iV_4$$

for all $i = 1, 2, \cdots, M$. The points (3) and coefficients (4) will then be

$$(6) \qquad \begin{array}{lll}
(\mu_{i1}, \mu_{i1}, \mu_{i2}, \mu_{i2}, 0, \cdots, 0)_S & B_{i,2,2} & = a_iV_n , \\
\left.\begin{array}{l}(\mu_{i1}, \mu_{i1}, \mu_{i2}, 0, 0, \cdots, 0)_S \\ (\mu_{i1}, \mu_{i2}, \mu_{i2}, 0, 0, \cdots, 0)_S\end{array}\right. & \left.\begin{array}{l}B_{i,2,1} \\ B_{i,1,2}\end{array}\right\} & = -(n - 4)a_iV_n , \\
\left.\begin{array}{l}(\mu_{i1}, \mu_{i1}, 0, 0, 0, \cdots, 0)_S \\ (\mu_{i2}, \mu_{i2}, 0, 0, 0, \cdots, 0)_S\end{array}\right. & \left.\begin{array}{l}B_{i,2,0} \\ B_{i,0,2}\end{array}\right\} & = \dfrac{(n - 3)(n - 4)}{2}\, a_iV_n , \\
(\mu_{i1}, \mu_{i2}, 0, 0, 0, \cdots, 0)_S & B_{i,1,1} & = (n - 3)(n - 4)a_iV_n , \\
\left.\begin{array}{l}(\mu_{i1}, 0, 0, 0, 0, \cdots, 0)_S \\ (\mu_{i2}, 0, 0, 0, 0, \cdots, 0)_S\end{array}\right. & \left.\begin{array}{l}B_{i,1,0} \\ B_{i,0,1}\end{array}\right\} & = \dfrac{-(n - 2)(n - 3)(n - 4)}{2}\, a_iV_n , \\
(\ 0\quad 0, 0, 0, 0, \cdots, 0) & B_{i,0,0} & = \dfrac{(n - 1)(n - 2)(n - 3)(n - 4)}{4}\, a_iV_n , \\
 & & i = 1, 2, \cdots, M .
\end{array}$$

Here $B_{i,0,0}$ is only part of the coefficient of the point $(0, 0, \cdots, 0)$ in the extended formula; this coefficient is $\sum_{i=1}^{M} B_{i,0,0}$.

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be fixed, but arbitrary, positive integers which satisfy

$$\begin{aligned} 0 &< \alpha_1 \leqq d\,, \\ 0 &< \alpha_1 + \alpha_2 \leqq d\,, \\ 0 &< \alpha_1 + \alpha_2 + \alpha_3 \leqq d\,, \\ 0 &< \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leqq d\,, \end{aligned}$$

where $d \leqq 9$. We show that formula (6) is exact for each of the five monomials

$$(7) \qquad x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}x_4^{\alpha_4}\,, \qquad x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\,, \qquad x_1^{\alpha_1}x_2^{\alpha_2}\,, \qquad x_1^{\alpha_1}\,, \qquad 1\,.$$

Consider, for example, the monomial $x_1^{\alpha_1}x_2^{\alpha_2}$. The assumption that (5) is a formula of degree d, $d \leqq 9$, implies that

$$V_m \sum_{i=1}^{M} a_i[\mu_{i1}^{\alpha_1}\mu_{i1}^{\alpha_2} + 2\mu_{i1}^{\alpha_1}\mu_{i2}^{\alpha_2} + 2\mu_{i1}^{\alpha_2}\mu_{i2}^{\alpha_1} + \mu_{i2}^{\alpha_1}\mu_{i2}^{\alpha_2}] = I_{C_m}(x_1^{\alpha_1}x_2^{\alpha_2})\,.$$

Using formula (6) to approximate $I_{C_n}(x_1^{\alpha_1}x_2^{\alpha_2})$ we can verify that we obtain

$$V_n \sum_{i=1}^{M} a_i[\mu_{i1}^{\alpha_1}\mu_{i1}^{\alpha_2} + 2\mu_{i1}^{\alpha_1}\mu_{i2}^{\alpha_2} + 2\mu_{i1}^{\alpha_2}\mu_{i2}^{\alpha_1} + \mu_{i2}^{\alpha_1}\mu_{i2}^{\alpha_2}]\,.$$

By the remark made at the end of Section 1 this shows that formula (6) is exact for $x_1^{\alpha_1}x_2^{\alpha_2}$. In a similar way we can verify that (6) is exact for all the monomials (7). By symmetry it follows that (6) is also exact for all monomials

$$x_{p_1}^{\alpha_1}x_{p_2}^{\alpha_2}x_{p_3}^{\alpha_3}x_{p_4}^{\alpha_4}\,, \qquad x_{p_1}^{\alpha_1}x_{p_2}^{\alpha_2}x_{p_3}^{\alpha_3}\,, \qquad x_{p_1}^{\alpha_1}x_{p_2}^{\alpha_2}\,, \qquad x_{p_1}^{\alpha_2}$$

where (p_1, p_2, p_3, p_4) is any permutation of $(1, 2, 3, 4)$.

To complete the proof that (6) has degree d there only remains to show that (6) is exact for all monomials of the form

$$(8) \quad x_{p_1}^{\alpha_1}x_{p_2}^{\alpha_2}\cdots x_{p_s}^{\alpha_s}\,, \qquad 4 < s \leqq n, \alpha_i > 0, i = 1, \cdots, s, 0 < \alpha_1 + \cdots + \alpha_s \leqq 9\,.$$

We note that in each monomial (8) the α_i cannot all satisfy $\alpha_i \geqq 2$, $i = 1, \cdots, s$. Therefore, for at least one i we must have $\alpha_i = 1$. This means that

$$I_{C_n}(x_{p_1}^{\alpha_1}x_{p_2}^{\alpha_2}\cdots x_{p_s}^{\alpha_s}) = 0\,.$$

But formula (6) also gives zero for the integral of (8) because each point of (6) has at most four nonzero coordinates.

In a similar way we can verify that if a formula (2) for C_4 consists of any collection of points and has degree $d \leqq 9$ then the extended formula (3) also has degree d.

3. An Example. Albrecht and Collatz [1] have given the following 5th-degree 7-point formula for C_2:

$$\begin{array}{ll} (0, 0) & 2V_2/7\,, \\ (r, r) & 25V_2/168\,, \\ (-r, -r) & 25V_2/168\,, \\ (s, -t)_S & 5V_2/48\,, \\ (-s, t)_S & 5V_2/48\,, \end{array}$$

$$r^2 = 7/15, \qquad s^2 = (7 + (24)^{1/2})/15\,, \qquad t^2 = (7 - (24)^{1/2})/15\,.$$

The extension of this formula gives the following 5th-degree formula for C_n which uses $3n^2 + 3n + 1$ points:

$$\begin{array}{ll} \pm(r, r, 0, \cdots, 0)_S & 25V_n/168\,, \\ \pm(r, 0, 0, \cdots, 0)_S & -25(n-2)V_n/168\,, \\ \pm(s, -t, 0, \cdots, 0)_S & 5V_n/48\,, \\ \pm(s, 0, 0, \cdots, 0)_S & -5(n-2)V_n/48\,, \\ \pm(t, 0, 0, \cdots, 0)_S & -5(n-2)V_n/48\,, \\ (0, 0, 0, \cdots, 0) & (5n^2 - 15n + 14)V_n/14\,. \end{array}$$

Here $\pm(r, r, 0, \cdots, 0)_S$ denotes the two sets of points $(r, r, 0, \cdots, 0)_S$ and $(-r, -r, 0, \cdots, 0)_S$.

4. Remarks. If formula (2) for C_m is fully-symmetric and if we denote it by $R^{(m)}$ as Lyness [3] does, then our extension of $R^{(m)}$ coincides with the formula denoted by Lyness as $E_m{}^n(0)R^{(m)}$. We have not discussed formulas which correspond to the $E_m{}^n(\gamma)R^{(m)}$, $\gamma \neq 0$, of Lyness.

The method described in Section 2 for extending a formula for C_m can also be applied to certain other special regions. Let R_1 be a one-dimensional region and $w_1(x) \geqq 0$ a weight function which satisfy $\int_{R_1} w_1(x)x^k dx = 0$, k an odd integer, $0 < k \leqq d$. Let $R_m = R_1 \times R_1 \times \cdots \times R_1$ and $w_m(x_1, \cdots, x_m) = w_1(x_1)\cdots w_1(x_m)$. Given a symmetric integration formula of degree $d \leqq 2m + 1$ for

$$\int_{R_m} \cdots \int w_m(x_1, \cdots, x_m) f(x_1, \cdots, x_m) dx_1 \cdots dx_m \tag{9}$$

we can extend this formula—by a method exactly similar to the method for C_m—to obtain a symmetric formula of degree d for

$$\int_{R_n} \cdots \int w_n(x_1, \cdots, x_n) f(x_1, \cdots, x_n) dx_1 \cdots dx_n\,.$$

As an example of (9) we have

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\,(-x_1{}^2 - \cdots - x_m{}^2) f(x_1, \cdots, x_m) dx_1 \cdots a x_m\,. \tag{10}$$

Lyness [4] has discussed extensions of fully-symmetric formulas for (10).

The method of extension discussed here (and by Lyness) has the undesirable property of producing integration formulas with both positive and negative coefficients. Hopefully, methods of extension will be found which do not introduce negative coefficients.

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