An Evaluation of Golomb's Constant

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1. Introduction. Solomon W. Golomb [1] has defined a constant λ which is related to the limiting properties of random permutations. Let L_N be the expected length of the longest cycle of a random permutation of N letters. Define $\lambda_N = L_N/N$. (Thus $\lambda_1 = 1$, $\lambda_2 = 3/4$, $\lambda_3 = 13/18$, $\lambda_4 = 67/96$.) It can be shown that the sequence $\{\lambda_N\}$ is monotonically decreasing, and thus a limit λ exists. Golomb has calculated $\lambda = 0.62432965 \cdots$. Golomb [2] poses the question of whether λ is related to any classical mathematical constants.

Shepp and Lloyd [3], using a different approach, calculated $\lambda = 0.62432997 \cdots$. Their value differs from Golomb's in the seventh and eighth digits. In this paper we consider more accurate computations on an IBM 7094, which allows us to calculate λ to 53-digit accuracy. It is expected that this result will make possible reliable comparison of λ with any classical mathematical constant.

2. Computational Formulas. There are two formulas known for λ. Golomb [1] derives the following:

$$\begin{aligned} &R(Y) = 1 \qquad \text{for } 1 \leqq Y \leqq 2\,, \\ &R'(Y) = -R(Y-1)/(Y-1) \qquad \text{for } 2 \leqq Y\,, \\ &\lambda = \int_1^\infty \frac{R(Y)}{Y^2}\,dY\,. \end{aligned} \tag{1}$$

This is equivalent to evaluating the limit of a system of two first-order differential equations. This is complicated by the fact that the mth derivative of R has a jump discontinuity at $Y = m + 1$. Consequently, higher-order derivatives are undefined at these points. Thus special care must be exercised in using numerical methods for solving these equations, particularly Newton-Cotes methods.

Shepp and Lloyd [3] derived a second formula:

$$\lambda = \int_0^\infty \exp\left[-E(x) - x\right]dx \quad \text{where} \quad E(x) = \int_x^\infty \frac{e^{-y}}{y}\,dy\,. \tag{2}$$

3. Computational Methods. For the present computation of λ, Golomb's method (1) was used. Shepp and Lloyd's method (2) was considered but rejected because it requires the evaluation of two power series ($E(x)$ and e^x) for each evaluation of the integrand. This is not feasible for the multiple-precision methods necessary for this paper. For less accuracy the method could be used with Hastings' [4] asymptotic formula for $E(x)$, which is accurate to $2 \cdot 10^{-8}$.

$R(Y)$ was evaluated by expressing it as a multiple-precision power series over intervals of the form $[X_0, X_0 + k]$. Thus

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$$R(Y) = R(X_0 + h) = \sum_{i=0}^{n} a_i h^i + O(h^{n+1}) . \tag{3}$$

If $X_0^* = X_0 - 1$, then (1) gives

$$\begin{aligned} R'(Y) = R'(X_0 + h) &= - R(X_0^* + h)/(X_0^* + h) \\ &= - \sum_{i=0}^{n} \frac{a_i^* h^i}{X_0^* + h} + O(k^{n+1}) \\ &= - \sum_{i=0}^{n} b_i^* h^i + O(k^{n+1}) \end{aligned} \tag{4}$$

where

$$b_0^* = \frac{1}{X_0^*} a_0^* , \qquad b_i^* = \frac{1}{X_0^*} (a_i^* - b_{i-1}^*) .$$

Thus

$$\begin{aligned} R(X_0 + h) &= R(X_0) - \sum_{i=0}^{n} \frac{b_i^*}{i + 1} h^{i+1} + O(k^{n+2}) \\ &= R(X_0) + \sum_{i=1}^{n} a_i h^i - \frac{b_n^*}{n + 1} h^{n+1} + O(k^{n+2}) \\ &= \sum_{i=0}^{n} a_i h^i + O(k^{n+1}) \end{aligned} \tag{5}$$

where

$$\begin{aligned} a_0 &= R[(X_0 - k) + k] + O(k^{n+1}) = \sum_{i=0}^{n} a_i' k^i , \\ a_i &= -b_{i-1}^*/i , \end{aligned}$$

and the a_i' are the coefficients for the power series for $R((X_0 - k) + h)$.

Similarly,

$$\begin{aligned} \int_{X_0}^{X_0+k} \frac{R(Y)}{Y^2} dY &= - \int_{X_0}^{X_0+k} \frac{R'(Y + 1)}{Y} dY \\ &= \int_{X_0}^{X_0+k} \sum_{i=0}^{n} \frac{b_i h^i + O(k^{n+1})}{X_0 + h} dh \\ &= \int_{0}^{k} \sum_{i=0}^{n} c_i h^i dh \\ &= \sum_{i=0}^{n} \frac{c_i}{i + 1} k^{i+1} \end{aligned} \tag{6}$$

where

$$c_0 = b_0/X_0 , \qquad c_i = (b_i - c_{i-1})/X_0 .$$

If X_0 is defined as $m/10$ for integer m and if $k = 1/10$, then the only multiple-precision operations required are addition, subtraction, division by a single-precision number, multiplication by 10, and division by powers of 10. This eliminates multiple-precision multipliers and divisors. These are the more difficult multiple-precision operations.

The discontinuous derivatives are no problem because the intervals $[m/10, m/10 + 1/10]$ contain integers only at the end points.

4. Results. A calculation of λ was made using 65-digit multiple-precision numbers, 52-term power series, and a range of integration from 1 to 33. The accuracy of this calculation depends upon each of these three factors.

The use of finite power series introduces error in three ways. First, there is the error resulting from termination of the series. Since the coefficients of each power series alternate in sign, this error is less than $a_{52} \cdot 10^{-52}$. Since a_{52} is positive, this error is positive. Second, there is the error in a_0 resulting from previous truncation errors. In this way, the error in each power series propagates to later series. Analysis of numerical results shows that the total accumulated error from both these factors is less than $2.4 \cdot 10^{-54}$. The effect of this error on λ is less than $1.2 \cdot 10^{-54}$. The third way in which truncation error enters is in the integration of $R'(Y + 1)/Y$. This error is roughly $1 \cdot 10^{-53}$. However, most of this error occurs in the interval (1, 2). If this portion of the error is eliminated by applying a correction so that the integral over (1, 2) is exact, then the error is less than $1 \cdot 10^{-54}$. This error is also positive, so the error in the evaluation of λ resulting from the termination of the power series is less than $2.2 \cdot 10^{-54}$.

Evaluation of the error induced by using a finite range of integration requires a maximizing function.

Let $S(Y) = \ln R(Y)$, then

$$S'(Y) = \frac{R'(Y)}{R(Y)} = -\frac{R(Y-1)}{R(Y)(Y-1)} < \frac{-1}{Y} \tag{7}$$

because $R(Y) > 0$ implies $R'(Y) < 0$ and thus $R(Y-1) > R(Y)$.

If $s(Y_0) = S(Y_0)$ and $s'(Y) = -1/Y$ then

$$S(Y) \leqq s(Y) = S(Y_0) + \ln Y_0 - \ln Y \quad \text{for} \quad Y_0 = Y\,. \tag{8}$$

Thus

$$R(Y) \leqq r(Y) = e^{s(Y)} = R(Y_0)Y_0/Y$$

and

$$\int_{Y_0}^{\infty} \frac{R(Y)}{Y^2}\,dY \leqq \int_{Y_0}^{\infty} \frac{r(Y)}{Y^2}\,dY = \frac{R(Y_0)}{2Y_0}\,. \tag{9}$$

Several values of R are given in Table 1. Fewer than five digits are given when truncation error casts doubt on succeeding digits. This evaluation of λ was carried out over the interval (1, 33). Since $R(33) = 1.22 \cdot 10^{-54}$, possible truncation error casts doubt on this value. Taking $Y_0 = 32$ gives a more reliable result, an error less than $2.1 \cdot 10^{-52}/2 \cdot 32 = 3.3 \cdot 10^{-54}$. However, the error is probably smaller by a factor of 80. This is derived by defining $s'(Y) = -c/Y$ where $c = R(Y_0 - 1)/R(Y_0)$. This ratio appears to decrease with Y, and thus $S(Y) \leqq s_c(Y)$. Then

$$\int_{Y_0}^{\infty} \frac{R(Y)}{Y^2}\,dY \leqq \int_{Y_0}^{\infty} \frac{r_c(Y)}{Y^2}\,dY = \frac{R(Y_0)}{(c+1)Y_0}. \tag{10}$$

Since 65 digits were used in multiple-precision numbers, simple analysis of computational methods shows that this factor can not produce an error as large as the other two factors. As a matter of fact, round-off error is probably less than $1 \cdot 10^{-63}$.

Thus we may conclude that

$$\lambda = 0.6243299885\ 4355087099\ 2936383100\ 8372441796\ 4262018052\ 9286 + \epsilon,$$

where $|\epsilon| \leqq 5.5 \cdot 10^{-54}$.

TABLE 1

Y	$R(Y)$
2	1.00000
3	0.30685
4	0.48608×10^{-1}
5	0.49193×10^{-2}
6	0.35472×10^{-3}
7	0.19650×10^{-4}
8	0.87457×10^{-6}
9	0.32321×10^{-7}
10	0.10162×10^{-8}
11	0.27702×10^{-10}
12	0.66448×10^{-12}
13	0.14197×10^{-13}
14	0.27292×10^{-15}
15	0.47606×10^{-17}
16	0.75899×10^{-19}
17	0.11129×10^{-20}
18	0.15091×10^{-22}
19	0.19014×10^{-24}
20	0.22354×10^{-26}
21	0.24618×10^{-28}
22	0.25480×10^{-30}
23	0.24864×10^{-32}
24	0.22937×10^{-34}
25	0.20055×10^{-36}
26	0.16658×10^{-38}
27	0.13173×10^{-40}
28	0.99361×10^{-43}
29	0.71621×10^{-45}
30	0.49418×10^{-47}
31	0.3269×10^{-49}
32	0.21×10^{-51}
33	0.1×10^{-53}

By using the smaller error estimate for the finite range of integration, we get $|\epsilon| \leqq 3 \cdot 10^{-54}$. We may then conclude that the 53rd digit of λ is 8. The 54th digit is in doubt.

This calculation required seven minutes on an IBM 7094.

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