# Explicit Inverses and Condition Numbers of Certain Circulants

By S. Charmonman and R. S. Julius

**Abstract.** Explicit inverses and condition numbers of two test-circulants with first rows $\{a, a + h, \cdots, a + (n - 1)h\}$ and $\{a, ah, \cdots, ah^{n-1}\}$ respectively are given in terms of the parameters defining the circulants.

**1. Introduction.** A circulant with the first row $(1, 2, \cdots, n)$ was used in testing inversion algorithms [6] but its explicit inverse was not given in the list of explicit inverses of some particular matrices in [5]. This paper gives explicit inverses and condition numbers of two circulants whose special cases include the one used in [6].

The simple result that the inverse of a circulant is a circulant [2] can be easily extended to the case of $r$-circulants. An $r$-circulant as defined in [4] is a square matrix of order $n$ in which the $i$th row, $i = 2, 3, \cdots, n$, is obtained from the $(i - 1)$th row by cyclically shifting each element $r$ places to the right. The word "row" can be replaced by the word "column" if "right" is replaced by "down." If we also say that shifting a negative number of places right means shifting left, then an $r$-circulant is also a $(kn + r)$-circulant, for any integer $k$.

Theorem 1.1. *The inverse of a nonsingular $r$-circulant $A$ is an $s$-circulant $B$ where $s$ satisfies*

$$rs = kn + 1 \tag{1.1}$$

*for some integer $k$.*

*Proof.* Let $e_1$ denote the first column of $I$. The set of equations

$$Af = e_1 \tag{1.2}$$

has a unique solution $f$, since $A$ is nonsingular. Let $B$ be the $s$-circulant with first column $f$, where $s$ satisfies (1.1). Then, by Theorem (3.1) in [1], $AB$ is an $(rs = kn + 1)$ circulant with first column $e_1$; that is, $AB = I$.

**2. Explicit Inverses.** Since the inverse of a circulant is a circulant we shall hypothesize that the inverse of a circulant which is defined by a few parameters can be explicitly expressed in terms of these parameters. The forms of the expressions can be conveniently observed from the results of numerical experiments on a digital computer and applications of (1.2) then give the required explicit inverses.

In the following two theorems $s$ is defined as in (1.1). $A_1$ and $A_2$ are nonsingular $r$-circulants of order $n \geqq 2$ with first row $\{a, a + h, \cdots, l = a + (n - 1)h\}$ for $A_1$, and $\{a, ah, \cdots, ah^{n-1}\}$ for $A_2$. $R_i(A)$ denotes the $i$th row, $C_i(A)$ the $i$th column and $a_{ij}$ the $(i, j)$th element of $A$.

Received May 18, 1967. Revised August 28, 1967.

THEOREM 2.1. *The inverse of* $A_1$ *is the s-circulant with the first column* $\{b - \alpha, b, \cdots, b, b + \alpha\}^T$, *where* $b = 2/\{n^2(a + l)\}$ and $\alpha = 1/(nh)$.

*Proof.* Let $y$ be the first column of an $s$-circulant $B$. According to (1.2) $B$ will be the inverse of $A$ if the $i$th element of

$$A_1 y = R_i(A)C_1(B) = b \sum_{i=1}^{n} [a + (i - 1)h] + \alpha(a_{in} - a_{i1})$$

is $e_1$. By observation of $A_1$ we have

$$\begin{aligned} a_{in} - a_{i1} &= (n - 1)h\,, \qquad & i = 1\,, \\ &= -h\,, & i \neq 1\,. \end{aligned}$$

Therefore, $A_1 y = e_1$ if $\alpha nh = 1$ and $bn(a + l)/2 - \alpha h = 0$ which give the required expressions for $b$ and $\alpha$.

Obviously Theorem 2.1 gives the inverse of a $(\pm 1)$-circulant with first row $(1, 2, \cdots, n)$ as the $(\pm 1)$-circulant with the first column $n^{-1}(b - 1, b, \cdots, b, b + 1)^T$ where $b = 2/\{n(n + 1)\}$.

THEOREM 2.2. *The inverse of* $A_2$ *is the s-circulant with first column* $\{b, 0, \cdots, 0, -hb\}^T$ *when* $b = 1/\{a(1 - h^n)\}$.

A proof can be easily constructed similar to that for Theorem 2.1 by using the property of $A_2$ that

$$\begin{aligned} ha_{in} - a_{i1} &= a(h^n - 1)\,, \qquad & i = 1\,, \\ &= 0\,, & i \neq 1\,. \end{aligned}$$

**3. Condition Numbers.** Circulants which are usually employed in testing numerical algorithms are the 1-circulant which are generally nonsymmetric and the $(-1)$-circulants which are always symmetric. One measure of the condition of these matrices, denoted here by $P(A)$, is the ratio of the largest (in modulus) to the smallest eigenvalue [7].

An expression for the eigenvalues of a 1-circulant may be found in [3] and [5]:

$$\lambda_s = \sum_{j=1}^{n} a_j t_s^{\,j-1} \tag{3.1}$$

where $t_s = \cos(2\pi s/n) + i \sin(2\pi s/n)$, $s = 1, 2, \cdots, n$, and $a_j$ is the $j$th element of the first row of the circulant.

An expression for the $n$ eigenvalues of a $(-1)$-circulant does not seem to be readily available in the literature but can be easily shown to be

$$\lambda_0;\ \pm (\lambda_s \lambda_{n-s})^{1/2}\,, \qquad s = 1, 2, \cdots, (n - 1)/2\,, \tag{3.2}$$

for odd $n$. The $\lambda_s$ are the eigenvalues of the 1-circulant whose first row is the same as the $(-1)$-circulant in question. When $n$ is even the set of $n$ eigenvalues of the $(-1)$-circulant are $\lambda_{(n/2)}$ and those in (3.3) with $s = 1, 2, \cdots, (n - 2)/2$.

Examinations of the sets of eigenvalues of the $(\pm 1)$-circulants of the forms $A_1$ and $A_2$ did not reveal expressions for their condition numbers. Since the explicit inverses of $A_1$ and $A_2$ as given in Theorems 2.1 and 2.2 are simpler than the matrices themselves, eigenvalues of the inverses were examined. After algebraic and trigonometric manipulations, we have for both the 1-circulants and $(-1)$-circulants,

$$P(A_1) \sim n + 2a/h\,, \qquad h > 0\,, \tag{3.4}$$

and

$$P(A_2) \sim \text{Max}\,[p_2 = |1 + h|/|1 - h|,\, 1/p_2]\,, \tag{3.5}$$

for large $n$. The symbol "$\sim$" is read "asymptotically equals." In the special case in which $a = h = 1$ in $A_1$, (3.4) gives $P(A_1) \sim n$ in agreement with the result in [5].

**4. Acknowledgement.** The preparation of this paper was supported by the National Research Council of Canada Grant No. NRC A-4076.

Department of Computing Science,
University of Alberta,
Edmonton, Alberta, Canada.

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