A Note on a Maximum Principle for the DuFort-Frankel Difference Equation

By Paul Gordon

Consider the parabolic partial differential equation

$$\partial u/\partial t = \sigma\, \partial^2 u/\partial x^2 \tag{1}$$

where σ is a positive constant.

Suppose initial and boundary conditions are given as follows:

$$\begin{aligned} u(0, x) &= f_1(x) : \quad 0 \leqq x \leqq x_1 , \\ u(t, 0) &= f_2(t) : \quad 0 \leqq t \leqq t_1 , \\ u(t, x_1) &= f_3(t) : \quad 0 \leqq t \leqq t_1 . \end{aligned} \tag{2}$$

Suppose that in the region $0 \leqq t \leqq t_1$, $0 \leqq x \leqq x_1$, this data determines a continuously differentiable solution, $u(t, x)$, of Eq. (1). Let

$$m = \max_{x,\, t} [|f_1(x)|, |f_2(t)|, |f_3(t)|] . \tag{3}$$

It is well known that $u(t, x)$ satisfies the following boundedness property:

$$|u(t, x)| \leqq m . \tag{4}$$

A difference equation representation for Eq. (1) would be expected, if it is to be convergent, to satisfy some kind of a bound similar to Eq. (3). The usual explicit and implicit difference equations satisfy precisely this bound [3, p. 13 and p. 47]. It is also well known that the DuFort-Frankel scheme satisfies some kind of a maximum principle. If one works with the L_2-norm, the form of the bound is quite clear [3, p. 83]. With respect to the maximum norm, it is also known that a maximum principle holds [2, p. 127], but its form is somewhat obscure. The purpose of this note is to derive the maximum principle satisfied by the DuFort-Frankel scheme in a relatively elementary fashion and to exhibit the dependence of this bound on the initial data.

The DuFort-Frankel difference equation can be written as follows:

$$(1 + q)U_j^{n+1} = (1 - q)U_j^{n-1} + q(U_{j+1}^n + U_{j-1}^n) , \tag{5}$$

where

$$q = 2\sigma \Delta t/\Delta x^2 , \qquad U_j^n = U(n\Delta t, j\Delta x) .$$

Let us suppose that Δx is specified as some function of Δt, $\Delta x = \Delta x(\Delta t)$. The consistency condition [3, p. 83] requires

$$\lim_{\Delta t = 0} (\Delta t/\Delta x) = 0 . \tag{6}$$

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Instead of proceeding in the time direction, the trick we employ is to suppose that the calculations proceed along the diagonals $x + t =$ constant. That is, at the Nth step obtain the values of U_j^n satisfying $n + i = N + 2$. This means that at the Nth step the following system of equations is to be solved:

$$(7) \qquad (1 + q)U_i^{N+2-i} - qU_{i+1}^{N+1-i} = (1 - q)U_i^{N-i} + qU_{i-1}^{N+1-i}, \qquad 1 \leqq i \leqq N.$$

(If any of the other boundaries are encountered by the diagonal, the system of equations is simply cut off appropriately.) It is assumed that U_j^0, U_j^1, U_0^n, $U_{x_1}^n$ are known from the data, Eq. (2), and that the same bound is satisfied.

$$(8) \qquad m = \max_{x,t} [|U_j^0|, |U_j^1|, |U_0^n|, |U_{x_1}^n|].$$

Let

$$(9) \qquad L_j^n = (1 - q)U_j^n + qU_{j-1}^{n+1}.$$

Then Eq. (7) can be solved as follows:

$$(10) \qquad (1 + q)U_{N+2-i}^i = q\left(\frac{q}{1+q}\right)^{i-2} U_{N+1}^1 + \sum_{\nu=0}^{i-2} \left(\frac{q}{1+q}\right)^{i-2-\nu} L_{N-\nu}^\nu.$$

Let

$$(11) \qquad \bar{L}_n = \max_\nu \{|U_{n+1}^1|, |L_{n-\nu}^\nu|\}.$$

Then,

$$(12) \qquad |U_{N+2-i}^i| \leqq \bar{L}_N.$$

It remains to obtain a bound for $\bar{L}_N$. From Eqs. (9) and (10), after some manipulation, we obtain the following:

$$(13) \qquad \begin{aligned} L_{n-i}^i = \frac{1}{q}\left(\frac{q}{1+q}\right)^i U_{n-1}^1 + \frac{1}{q^2}\left(\frac{q}{1+q}\right)^i \sum_{\nu=0}^{i-2}\left(\frac{1+q}{q}\right)^\nu L_{n-2-\nu}^\nu \\ + \left(\frac{q}{1+q}\right) L_{n-1-i}^{i-1}. \end{aligned}$$

Thus,

$$(14) \qquad \bar{L}_N \leqq \max [\bar{L}_{N-2}, |L_N^0|, |L_{N-1}^1|, |U_{N+1}^1|].$$

But L_N^0 and L_N^1 depend on the initial data. A simple series expansion shows that

$$|L_N^0| \leqq C(\Delta t/\Delta x) + |U_N^0|,$$

where C is determined by the data. The same holds for L_N^1. Thus,

$$(15) \qquad |U_j^n| \leqq m + C\frac{\Delta t}{\Delta x},$$

where $\Delta t/\Delta x$ satisfies Eq. (6). Equation (15) is now to be compared with Eq. (4).

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