Simplified Calculation of Ei(x) for Positive Arguments, and a Short Table of Shi(x)*

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Currently there does not seem to be an efficient computer subroutine for evaluating the real exponential integral

$$\text{(1)} \qquad \mathrm{Ei}\,(x) = -\int_{-x}^{\infty} \frac{e^{-t}\,dt}{t} = -E_1(-x)$$

(where the principal value is understood for positive values of the argument) for $x > 4$. Clenshaw, Miller and Woodger [2] give algorithms, based on Chebyshev series [1], for $x \leqq 4$, but the higher range is deferred for later investigation. The techniques of Harris [5], Kotani *et al.* [6], and Miller and Hurst [7] have been used successfully to compute tables of the function in this range, but are difficult to mechanize because they involve successive recursions from smaller values of the argument. In order to construct a minimax rational approximation, using for example the Remes algorithm as described by Cody and Stoer [3] or by Ralston [8], a procedure for computing the function to an accuracy somewhat greater than that desired in the approximation is required.

We found it is convenient to generate Ei (x) for $x > 0$ by using the relation

$$\text{(2)} \qquad E_1(x) + \mathrm{Ei}\,(x) = 2\,\mathrm{Shi}\,(x)\,,$$

where

$$\text{(3)} \qquad \begin{aligned} \mathrm{Shi}\,(x) &= \int_0^x \frac{\sinh t\,dt}{t} \\ &= \int_0^1 \frac{\sinh xt\,dt}{t}\,. \end{aligned}$$

We evaluated Ei (x) for $x = 0.5(0.5)30.0$ according to Eq. (2) using double-precision arithmetic on an IBM 7094 computer. Clenshaw's Chebyshev expansions [1] were used to generate $E_1(x)$. The function Shi (x) was evaluated by Gauss-Legendre quadrature, using the abscissas and weight factors given by Davis and Polonsky [4]. The values obtained for both $E_1(x)$ and Ei (x) agreed to 15S with Harris' tables [5]. To this accuracy, $E_1(x)$ is negligible in Eq. (3) for $x > 18$.

For $x > 30$, the improved asymptotic expansion of Wadsworth [9] is adequate to give Ei (x) to 15S.

Table 1 gives the values of Shi (x) for $x = 1.0(1.0)30.0$ to 13S, computed and verified as described above.

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TABLE 1. *A short table of* Shi (x)

x	Shi (x)			
1.0	1.05725	08753	76	
2.0	2.50156	74333	55	
3.0	4.97344	04758	60	
4.0	9.81732	69112	33	
5.0	2.00932	11825	70	(1)*
6.0	4.29950	61112	45	(1)
7.0	9.57524	29408	62	(1)
8.0	2.20189	96860	02	(2)
9.0	5.18939	15158	22	(2)
10.0	1.24611	44901	99	(3)
11.0	3.03570	31877	49	(3)
12.0	7.47976	63334	36	(3)
13.0	1.85988	44245	43	(4)
14.0	4.65962	56817	01	(4)
15.0	1.17477	92624	54	(5)
16.0	2.97780	49933	54	(5)
17.0	7.58318	94702	13	(5)
18.0	1.93895	21652	99	(6)
19.0	4.97545	36255	23	(6)
20.0	1.28078	26332	03	(7)
21.0	3.30635	93177	74	(7)
22.0	8.55723	35650	18	(7)
23.0	2.21983	18491	51	(8)
24.0	5.77057	69592	46	(8)
25.0	1.50297	54532	63	(9)
26.0	3.92147	04959	49	(9)
27.0	1.02482	48559	94	(10)
28.0	2.68225	59296	16	(10)
29.0	7.02995	97879	20	(10)
30.0	1.84486	60470	36	(11)

* The numbers in parentheses denote the power of 10 by which the corresponding entries are to be multiplied.

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