Chebyshev Approximations for the Fresnel Integrals*

By W. J. Cody

Abstract. Rational Chebyshev approximations have been computed for the Fresnel integrals $C(x)$ and $S(x)$ for arguments in the intervals [0., 1.2] and [1.2, 1.6], and for the related functions $f(x)$ and $g(x)$ for the intervals [1.6, 1.9], [1.9, 2.4] and [2.4, ∞]. Maximal relative errors range down to 2×10^{-19}.

1. Introduction. The Fresnel integrals are defined [1] by

$$C(x) - iS(x) = \int_0^x i^{-t^2} dt = \int_0^x \exp(-i\pi t^2/2) dt \,. \tag{1}$$

For small arguments the usual Taylor series expansions are quite useful computationally, while for large arguments the forms

$$C(x) = \tfrac{1}{2} + f(x) \sin(\pi x^2/2) - g(x) \cos(\pi x^2/2) \tag{2a}$$

and

$$S(x) = \tfrac{1}{2} - f(x) \cos(\pi x^2/2) - g(x) \sin(\pi x^2/2) \,, \tag{2b}$$

where $f(x)$ and $g(x)$ have well-known asymptotic expansions, are most useful. For values of x greater than that corresponding to the first maximum function value (in the vicinity of $x = 1$), evaluation of the Taylor series is subject to loss of accuracy through subtraction error. Since the asymptotic forms are not useful for small $|x|$, there is a region for which accurate evaluation of the Fresnel integrals is difficult.

In recent years a number of papers presenting approximations have appeared [2]–[4]. All of the approximations given are of somewhat limited usefulness, however. The Chebyshev series expansions given by Németh [4] converge painfully slowly, while the single approximation given by Boersma [3] is of limited accuracy and subject to subtraction error during evaluation. The approximation forms used by Syrett and Wilson [2] are generally quite inefficient and involve awkward transformations of variable. None of the previous investigators has considered approximation by rational functions, although such approximations are generally more efficient than pure polynomial approximations. It is our purpose to present efficient rational approximations for $S(x)$ and $C(x)$ when $|x|$ is small, and for $f(x)$ and $g(x)$ for all other x, with maximal relative errors ranging down to 10^{-19} in some cases.

2. Approximation Forms. The approximation forms and intervals used are:

Received March 1, 1967. Revised October 9, 1967.

* Work performed under the auspices of the U. S. Atomic Energy Commission.

$$(3)\qquad \begin{aligned} C_{lm}(x) &= xR_{lm}(x^4) \\ S_{lm}(x) &= x^3R_{lm}(x^4) \end{aligned} \quad \text{for} \quad |x| \leqq 1.2 \text{ and } 1.2 \leqq |x| \leqq 1.6\ ;$$

$$(4)\qquad \begin{aligned} f_{lm} &= x^{-1}R_{lm}(x^{-4}) \\ g_{lm} &= x^{-3}R_{lm}(x^{-4}) \end{aligned} \quad \text{for} \quad 1.6 \leqq |x| \leqq 1.9 \text{ and } 1.9 \leqq |x| \leqq 2.4\ ;$$

and

$$(5)\qquad \begin{aligned} f_{lm}(x) &= x^{-1}\{1/\pi + x^{-4}R_{lm}(x^{-4})\} \\ g_{lm}(x) &= x^{-3}\{1/\pi^2 + x^{-4}R_{lm}(x^{-4})\} \end{aligned} \quad \text{for} \quad 2.4 \leqq |x|,$$

where R_{lm} is a rational function of degree l in the numerator and m in the denominator. The forms (3) and (5) are based upon the Taylor series and asymptotic forms of the functions involved, while the forms (4) are based upon the results of much experimentation.

The choice of the intervals of approximation again resulted from experimentation. While they are not optimal in this respect, the intervals were chosen so that a given choice of degree of numerator and denominator would result in roughly the same accuracy for each interval.

3. Computations. All computations were carried out in 25-significant figure arithmetic on a CDC 3600 computer. The approximations were computed using the Remes algorithm for rational Chebyshev approximations [5], [6]. Because the approximation forms used for large $|x|$ correctly emulate the asymptotic behaviour of $f(x)$ and $g(x)$, the error of approximation vanishes as $|x| \to \infty$. Consequently these approximations could be computed for large (but finite) upper limits to the interval of approximation.

Function values were computed as needed using the Taylor series expansions of $C(x)/x$ and $S(x)/x^3$ for $|x| \leqq 2.5$, and using the most accurate approximations given by Syrett and Wilson [2] for $2.5 \leqq |x| \leqq 4.0$. For $|x| > 4.0$ the asymptotic expansions for $f(x)$ and $g(x)$ were converted into continued fractions by means of the *QD* algorithm [7]. The function routines were extensively checked against the excellent tables of Syrett and Wilson, and against each other in slightly overlapping regions. These tests indicated an accuracy of 20S in the master functions.

The relative error curves

$$(6)\qquad \delta_{lm}(x) = (A(x) - A_{lm}(x))/A(x)$$

where A refers to C, S, f, or g, were all levelled to three significant figures. In addition each approximation, with the coefficients rounded as they appear in the tables, was tested against the master routines for 5000 pseudo-random arguments. In all cases maximal errors agreed (within roundoff) in magnitude and location with those given by the error curves (6) in the Remes algorithm.

4. Results. All results are given in tabular form in the microfiche supplement to this issue of the journal. Tables I–IV list the values of

$$E_{lm} = -100 \log_{10} \max |\delta_{lm}|$$

where the maximum is taken over the appropriate interval, for the initial segments of the various L_∞ Walsh arrays. An examination of the tables indicates E_{lm} is generally quite close to maximal for fixed $l + m$ along the line $l = m$. Tables V–VIII present the coefficients for the cases $l = m$. All coefficients are given to an accuracy greater than that justified by the maximal errors, but reasonable additional rounding should not greatly affect the overall accuracies.

Not all of the approximations have been checked for numerical stability of evaluation, but most of those that were checked proved to be quite stable numerically when the numerator and denominator polynomials were evaluated by nested multiplication. The few exceptions all occurred for the approximations for $S(x)$ and $C(x)$ over the interval $1.2 \leqq |x| \leqq 1.6$, when large subtraction errors occurred. Transforming the numerator and denominator polynomials into their equivalent finite Chebyshev polynomial expansions and using the Clenshaw-Rice scheme [8] for evaluating these gave great numerical stability with the usual penalty in speed of evaluation.

For $|x| \geqq 1.6$, Eqs. (2) must be used to compute $S(x)$ and $C(x)$. If we denote by $\Delta\beta$ the absolute error in the quantity β, and by $\delta\beta$ the relative error, i.e., $\delta\beta = \Delta\beta/\beta$, then we find from (2)

$$\delta C(x) \approx \frac{f(x)\sin(u)}{C(x)}\,\delta f(x) - \frac{g(x)\cos(u)}{C(x)}\,\delta g(x) + [f(x)\cos(u) + g(x)\sin(u)]\frac{\pi x}{C(x)}\,\Delta x\,,$$

where $u = \pi x^2/2$. If we assume $\Delta x = 0$, we find the direct contribution of $|\delta f(x)|$ and $|\delta g(x)|$ to $|\delta C(x)|$ is

$$|\delta C(x)| \leqq \frac{\max|f(x)|}{\min|C(x)|}\,|\delta f(x)| + \frac{\max|g(x)|}{\min|C(x)|}\,|\delta g(x)| = B|\delta f(x)| + C|\delta g(x)|\,.$$

Similarly,

$$|\delta S(x)| \leqq \frac{\max|f(x)|}{\min|S(x)|}\,|\delta f(x)| + \frac{\max|g(x)|}{\min|S(x)|}\,|\delta g(x)| = D|\delta f(x)| + E|\delta g(x)|\,.$$

Using the tables in [1] rough upper bounds on B, C, D, and E are easily computed for the intervals of interest to us, and are presented in Table IX. These bounds will be useful in deciding upon the accuracies necessary for the approximations for $f(x)$ and $g(x)$ in order to obtain a desired accuracy in the computation of $C(x)$ or $S(x)$. We note in particular that generally an error in $g(x)$ ten times as large as that in $f(x)$ can be tolerated for a given computation of $C(x)$ or $S(x)$.

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TABLE II

$$E_{\ell m} = -100 \log_{10} \left\| \frac{S(x) - S_{\ell m}(x)}{S(x)} \right\|_{\infty}$$

$|x| \leq 1.2$

m \ ℓ	0	1	2	3	4	5	6	
0	73	211	375	559	761	976	1203	1[illegible]
1		383	582	794	1018	1253	1497	1[illegible]
2			776	1014	1259	1511	1771	
3				1229	1494	1763		
4					1726			

$1.2 \leq |x| \leq 1.6$

m \ ℓ	0	1	2	3	4	5	6	
0	40	138	265	413	579	759	952	1[illegible]
1		270	433	610	800	1000	1210	1[illegible]
2			588	793	1005	1223	1450	
3				972	1204	1440		
4					1401	1653		
5						1866		

TABLE III

$$E_{\ell m} = -100 \log_{10} \left\| \frac{f(x) - f_{\ell m}(x)}{f(x)} \right\|_{\infty}$$

$1.6 \leq x \leq 1.9$

m \ ℓ	0	1	2	3	4	5	6
0	212	371	506	627	741	850	956
1		542	687	821	945	1063	1177
2		692	844	986	1119	1246	1367
3			990	1138	1278	1411	
4				1281	1427		
5						1713	

$1.9 \leq |x| \leq 2.4$

	0	1	2	3	4	5	6
0	226	391	527	648	759	864	964
1		558	701	830	949	1062	1170
2		704	850	986	1112	1232	1346
3			988	1128	1260	1385	
4			1116	1262	1398		
5						1664	

$2.4 \leq |x|$

	0	1	2	3	4	5	6
0	391	526	638	736	822	902	975
1	551	680	792	891	980	1062	1138
2	662	806	917	1018	1109	1194	
3	756	908	1029	1130	1223		
4		998	1124	1232	1326	1414	
5				1322	1422	1512	
6							1689

TABLE IV

$$E_{\ell m} = -100 \log_{10} \left\| \frac{g(x) - g_{\ell m}(x)}{g(x)} \right\|_{\infty}$$

$1.6 \leq |x| \leq 1.9$

m \ ℓ	0	1	2	3	4	5	6
0	151	291	415	531	640	745	847
1		465	602	729	849	963	1073
2		617	736	899	1027	1150	1268
3			910	1053	1189	1319	
4				1198	1340		
5						1625	

$1.9 \leq |x| \leq 2.4$

m \ ℓ	0	1	2	3	4	5	6
0	161	306	431	545	651	751	848
1		474	608	730	845	954	1058
2		620	759	889	1011	1126	1237
3			898	1033	1161	1282	
4			1024	1168	1301		
5						1565	

$2.4 \leq |x|$

m \ ℓ	0	1	2	3	4	5	6
0	300	422	525	616	697	772	841
1	455	574	678	771	855	933	1005
2	559	699	804	898	985	1066	
3	646	796	914	1010	1099		
4		881	1006	1112	1202	1287	
5			1088	1199	1298	1384	
6							1558

TABLE VA

$$C(x) \simeq x \sum_{s=0}^{n} p_s x^{4s} \Big/ \sum_{s=0}^{n} q_s x^{4s} \quad , \quad |x| \le 1.2$$

n = 1

s	p_s		q_s
0	1.00053	(00)	1.00000
1	-1.12353	(-01)	1.38937

n = 2

s	p_s		q_s
0	9.99999 896	(-01)	1.00000 000
1	-1.63090 954	(-01)	8.36467 414
2	1.06388 604	(-02)	3.10155 884

n = 3

s	p_s		q_s
0	1.00000 00000 042	(00)	1.00000 00000 0
1	-1.86511 27631 106	(-01)	6.02288 33908 1
2	1.50656 63274 457	(-02)	1.74103 00558 1
3	-3.10580 74693 185	(-04)	2.63086 17899 2

n = 4

s	p_s		q_s
0	9.99999 99999 99999 421	(-01)	1.00000 00000 0
1	-1.99460 89882 61842 706	(-01)	4.72792 11201 0
2	1.76193 95254 34914 045	(-02)	1.09957 21502 5
3	-5.28079 65137 26226 960	(-04)	1.55237 88527 6
4	5.47711 38568 26871 660	(-06)	1.18938 90142 2

TABLE VB

$$C(x) \cong x \sum_{s=0}^{n} p_s x^{4s} \Big/ \sum_{s=0}^{n} q_s x^{4s} \quad ; \quad 1.2 \leq |x| \leq 1.$$

n = 1

s	p_s		q_s
0	1.3139	(00)	1.0000
1	-5.8827	(-02)	4.7324

n = 2

s	p_s		q_s
0	9.97908 90	(-01)	1.00000 00
1	-1.53918 95	(-01)	8.98786 47
2	9.98179 33	(-03)	5.60030 71

n = 3

s	p_s		q_s
0	1.00000 44010 9	(00)	1.00000 00000 0
1	-1.83413 85190 8	(-01)	6.33347 44563 7
2	1.45776 17082 8	(-02)	2.01245 73836 9
3	-2.78980 70527 0	(-04)	4.03949 00464 6

n = 4

s	p_s		q_s
0	9.99999 99664 96876	(-01)	1.00000 00000 0
1	-1.98030 09870 22688	(-01)	4.87100 03089 9
2	1.73551 87484 50023	(-02)	1.18840 69740 0
3	-5.06944 22979 35788	(-04)	1.82452 76358 4
4	5.05996 22546 78234	(-06)	1.67831 42578 7

n = 5

s	p_s		q_s
0	1.00000 00000 01110 43640	(00)	1.00000 00000 0
1	-2.07073 36033 53238 94245	(-01)	3.96667 49695 2
2	1.91870 27943 17469 26505	(-02)	7.88905 24505 2
3	-6.71376 03469 49221 09230	(-04)	1.01344 63086 6
4	1.02365 43505 61058 64908	(-05)	8.77945 37789 2
5	-5.68293 31012 18707 28343	(-08)	4.41701 37406 5

TABLE VIA

$$S(x) \approx x^3 \sum_{s=0}^{n} p_s x^{4s} \Big/ \sum_{s=0}^{n} q_s x^{4s} \quad , \quad |x| \le 1.2$$

n = 1

s	p_s		q_s
0	5.23677	(-01)	1.00000
1	-4.67900	(-02)	8.81622

n = 2

s	p_s		q_s
0	5.23598 7665	(-01)	1.00000 0000
1	-5.83776 3961	(-02)	6.47494 4765
2	2.16592 0196	(-03)	1.71328 7588

n = 3

s	p_s		q_s
0	5.23598 77559 8566	(-01)	1.00000 00000 0
1	-6.59149 58113 9046	(-02)	5.03546 38867 0
2	3.21501 64982 8293	(-03)	1.17835 98035 6
3	-4.88704 43624 0178	(-05)	1.37089 87582 6

n = 4

s	p_s		q_s
0	5.23598 77559 82988 7021	(-01)	1.00000 00000 0
1	-7.07489 91514 45230 2596	(-02)	4.11223 15114 2
2	3.87782 12346 36828 7939	(-03)	8.17091 94215 2
3	-8.45557 28435 27768 0591	(-05)	9.62690 87593 9
4	6.71748 46662 51408 6196	(-07)	5.95281 22767 8

TABLE VIB

$$S(x) \simeq x^3 \sum_{s=0}^{n} p_s x^{4s} \Big/ \sum_{s=0}^{n} q_s x^{4s} \quad , \quad 1.2 \le |x| \le 1.6$$

n = 1

s	p_s		q_s
0	5.4766	(-01)	1.0000
1	-3.7151	(-02)	1.4559

n = 2

s	p_s		q_s
0	5.23439 94	(-01)	1.00000 00
1	-5.48283 47	(-02)	7.11047 04
2	1.90208 81	(-03)	2.55962 74

n = 3

s	p_s		q_s
0	5.23599 04049 8	(-01)	1.00000 00000 0
1	-6.46593 39242 6	(-02)	5.27536 68168 5
2	3.08030 79436 1	(-03)	1.34311 02682 1
3	-4.40800 85441 8	(-05)	1.90476 61284 9

n = 4

s	p_s		q_s
0	5.23598 77543 50917 8	(-01)	1.00000 00000 0000
1	-7.01490 76634 83366 2	(-02)	4.22680 67370 3954
2	3.80315 81605 98703 8	(-03)	8.76427 53831 0732
3	-8.09649 48714 40815 6	(-05)	1.10885 42889 7892
4	6.19080 80210 05277 2	(-07)	7.87258 29545 4784

= 5

s	p_s		q_s
0	5.23598 77559 83441 65913	(-01)	1.00000 00000 0000
1	-7.37766 91401 01913 23867	(-02)	3.53398 34216 7472
2	4.30730 52650 43665 10217	(-03)	6.18224 62019 5473
3	-1.09540 02391 14349 94566	(-04)	6.87086 26571 8620
4	1.28531 04374 27248 20610	(-06)	5.03090 58124 6612
5	-5.76765 81559 30888 04567	(-09)	2.05539 12445 8579

TABLE VIIA

$$f(x) \cong x^{-1} \sum_{s=0}^{n} p_s x^{-4s} \Big/ \sum_{s=0}^{n} q_s x^{-4s} \quad , \; 1.6 \leq |x| \leq 1.9$$

n = 1

s	p_s		q_s
0	3.18035 19	(-01)	1.00000 00
1	4.23523 32	(-01)	1.60154 03

n = 2

s	p_s		q_s
0	3.18285 25209 4	(-01)	1.00000 00000 0
1	2.02860 27571 3	(00)	6.67171 54813 5
2	1.08108 13604 8	(00)	4.52997 55397 2

n = 3

s	p_s		q_s
0	3.18306 46311 448	(-01)	1.00000 00000 000
1	4.60772 49266 684	(00)	1.47784 64938 936
2	1.19145 78438 074	(01)	4.09029 71163 705
3	3.55550 73665 830	(00)	1.61304 32784 819

n = 4

s	p_s		q_s
0	3.18309 26850 49059 9	(-01)	1.00000 00000 0000
1	8.03588 12280 39415 6	(00)	2.55491 61843 5795
2	4.80340 65557 79248 7	(01)	1.57611 00558 0122
3	6.98534 26160 10206 5	(01)	2.49561 99380 5172
4	1.35304 23554 03878 4	(01)	6.55630 64008 3915

n = 5

s	p_s		q_s
0	3.18309 75293 58098 5290	(-01)	1.00000 00000 0000
1	1.22260 00551 67296 1219	(01)	3.87130 03365 5834
2	1.29248 86131 90165 7025	(02)	4.16743 59830 7056
3	4.38863 67156 69554 7655	(02)	1.47400 30733 9666
4	4.14667 22177 95896 1672	(02)	1.53716 75584 8957
5	5.67714 63664 18511 6454	(01)	2.91130 88788 8478

TABLE VIIB

$$f(x) \stackrel{\sim}{=} x^{-1} \sum_{s=0}^{n} p_s x^{-4s} \Big/ \sum_{s=0}^{n} q_s x^{-4s} \quad , \quad 1.9 \le |x| \le 2.4$$

n = 1

s	p_s		q_s
0	3.18251 12	(-01)	1.00000 00
1	5.83959 51	(-01)	2.12439 44

n = 2

s	p_s		q_s
0	3.18306 99932	(-01)	1.00000 00000
1	2.99934 57087	(00)	9.72545 17756
2	2.39563 40010	(00)	9.48140 77696

n = 3

s	p_s		q_s
0	3.18309 63944 521	(-01)	1.00000 00000 000
1	7.07704 31878 327	(00)	2.25369 94405 207
2	2.87560 83262 903	(01)	9.61275 68469 278
3	1.35836 85742 326	(01)	5.74070 28004 031

n = 4

s	p_s		q_s
0	3.18309 85686 40159	(-01)	1.00000 00000 0000
1	1.26512 94696 83175	(01)	4.00491 53027 8100
2	1.21946 76164 98339	(02)	3.94205 96979 5158
3	2.91003 36555 12762	(02)	1.00136 84034 9569
4	9.27839 78286 31516	(01)	4.14267 62242 2243

n = 5

s	p_s		q_s
0	3.18309 88182 20169 217	(-01)	1.00000 00000 0000
1	1.95883 94102 19691 002	(01)	6.18427 13817 2887
2	3.39837 13492 69842 400	(02)	1.08535 06750 0650
3	1.93007 64078 67157 531	(03)	6.33747 15585 1143
4	3.09145 16157 44296 552	(03)	1.09334 24898 8808
5	7.17703 24936 51399 590	(02)	3.36121 69918 0551

TABLE VIIC

$$f(x) \approx x^{-1}\left\{\pi^{-1} + x^{-4} \sum_{s=0}^{n} p_s x^{-4s} \Big/ \sum_{s=0}^{n} q_s x^{-4s}\right\} , \quad 2.4$$

n = 0

s	p_s		q_s
0	-8.97969	(-02)	1.00000

n = 1

s	p_s		q_s
0	-9.67122 165	(-02)	1.00000 000
1	-3.73565 920	(-01)	7.29466 572

n = 2

s	p_s		q_s
0	-9.67541 44364 8	(-02)	1.00000 00000 0
1	-2.26566 29381 8	(00)	2.69596 93997 7
2	-3.53429 82108 4	(00)	9.72883 95746 9

n = 3

s	p_s		q_s
0	-9.67545 96389 025	(-02)	1.00000 00000 000
1	-5.52549 43840 897	(00)	6.06544 83020 979
2	-5.86062 82086 171	(01)	7.85288 41711 294
3	-5.22355 60918 394	(01)	1.85924 98578 831

n = 4

s	p_s		q_s
0	-9.67546 03169 52504	(-02)	1.00000 00000 000
1	-1.02387 64281 29288	(01)	1.09368 22440 534
2	-2.71257 96340 37998	(02)	3.15584 34619 205
3	-1.76611 93452 82127	(03)	2.62563 43162 044
4	-1.04346 44266 56267	(03)	4.57825 20572 463

TABLE VIIC - CONTINUED

n = 5

s	p_s		q_s
0	-9.67546 03296 70903 80	(-02)	1.00000 00000 0000
1	-1.64797 71284 12457 67	(01)	1.73871 69067 3649
2	-8.16343 40178 43745 98	(02)	9.01827 59623 1524
3	-1.34922 02817 18572 48	(04)	1.65946 46262 1853
4	-6.13547 11361 46997 72	(04)	1.00105 47890 0791
5	-2.61294 75322 51417 79	(04)	1.37012 36481 7225

n = 6

s	p_s		q_s
0	-9.67546 03299 52532 343	(-02)	1.00000 00000 0000
1	-2.43127 54071 94161 683	(01)	2.54828 90129 4973
2	-1.94762 19983 06889 176	(03)	2.09976 15368 5781
3	-6.05985 21971 60773 639	(04)	6.92412 25098 2770
4	-7.07680 69528 37779 823	(05)	9.17882 32299 1814
5	-2.41765 67490 61154 155	(06)	4.29273 32556 3018
6	-7.83491 45900 78317 336	(05)	4.80329 47842 6052

TABLE VIIIA

$$g(x) \simeq x^{-3} \sum_{s=0}^{n} p_s x^{-4s} \Big/ \sum_{s=0}^{n} q_s x^{-4s} \quad , \quad 1.6 \le |x| \le 1.9$$

n = 1

s	p_s		q_s
0	1.00701 1	(-01)	1.00000 0
1	1.45778 0	(-01)	2.70025 3

n = 2

s	p_s		q_s
0	1.01250 0483	(-01)	1.00000 0000
1	7.73520 7446	(-01)	9.10195 6178
2	3.87828 2770	(-01)	1.00223 4185

n = 3

s	p_s		q_s
0	1.01309 54368 17	(-01)	1.00000 00000 00
1	1.73193 79841 73	(00)	1.86007 74300 76
2	5.03658 82452 65	(00)	6.90195 19357 25
3	1.29070 72465 07	(00)	4.30891 86599 89

n = 4

s	p_s		q_s
0	1.01318 80965 09180	(-01)	1.00000 00000 0000
1	2.96622 05547 25899	(00)	3.07917 87367 2404
2	2.01543 52995 05393	(01)	2.36287 43189 8047
3	3.15560 56793 87908	(01)	4.95386 12582 4833
4	4.91601 28306 46366	(00)	2.02614 44934 0359

n = 5

s	p_s		q_s
0	1.01320 61881 02747 985	(-01)	1.00000 00000 0000
1	4.44533 82755 05123 778	(00)	4.53925 01967 3689
2	5.31122 81348 09894 481	(01)	5.83590 57571 6429
3	1.99182 81867 89025 318	(02)	2.54473 13318 1822
4	1.96232 03797 16626 191	(02)	3.48112 14785 6545
5	2.05421 43249 85006 303	(01)	1.01379 48339 6002

TABLE VIIIB

$$g(x) \simeq x^{-3} \sum_{s=0}^{n} p_s x^{-4s} \Big/ \sum_{s=0}^{n} q_s x^{-4s} \quad , \quad 1.9 \le |x| \le$$

n = 1

s	p_s		q_s
0	1.01171 1	(-01)	1.00000 0
1	2.23963 0	(-01)	3.60923 7

n = 2

s	p_s		q_s
0	1.01311 5463	(-01)	1.00000 0000
1	1.18202 1191	(00)	1.31721 9285
2	9.88631 5969	(-01)	2.10110 4994

n = 3

s	p_s		q_s
0	1.01320 20256 53	(-01)	1.00000 00000 0
1	2.69076 70137 70	(00)	2.80745 70055 0
2	1.28362 47492 71	(01)	1.59870 43135 2
3	5.79133 75877 23	(00)	1.52563 03850 4

n = 4

s	p_s		q_s
0	1.01321 05094 09046	(-01)	1.00000 00000 0
1	4.68276 97697 57399	(00)	4.77365 29667 0
2	5.24135 16133 46472	(01)	5.80203 69679 4
3	1.41173 59445 50041	(02)	1.94717 93272 4
4	4.01975 60127 88710	(01)	1.26604 12369 6

n = 5

s	p_s		q_s
0	1.01321 16176 18045 86	(-01)	1.00000 00000 0
1	7.11205 00178 97828 23	(00)	7.17128 59693 9
2	1.40959 61791 13155 24	(02)	1.49051 92279 7
3	9.08311 74952 95939 38	(02)	1.06729 67803 0
4	1.59268 00608 53538 64	(03)	2.41315 56721 3
5	3.13330 16306 87559 50	(02)	1.15149 83237 6

TABLE VIIIC

$$g(x) \cong x^{-3}\left\{\pi^{-2} + x^{-4} \Big/ \sum_{s=0}^{n} p_s x^{-4s} \quad \sum_{s=0}^{n} q_s x^{-4s}\right\} \quad , \quad 2.4$$

s	p_s		q_s
	n = 0		
0	-1.3549	(-01)	1.0000
	n = 1		
0	-1.53828 24	(-01)	1.00000 00
1	-6.45474 64	(-01)	1.02903 28
	n = 2		
0	-1.53987 60302	(-01)	1.00000 00000
1	-4.27728 57997	(00)	3.41499 86477
2	-6.59401 81141	(00)	1.70717 08816
	n = 3		
0	-1.53989 69716 16	(-01)	1.00000 00000 0
1	-1.02463 83144 46	(01)	7.29222 93037 5
2	-1.25250 74021 20	(02)	1.18652 86732 4
3	-1.02918 96761 44	(02)	3.84443 09084 7
	n = 4		
0	-1.53989 73305 7971	(-01)	1.00000 00000 0
1	-1.86483 22383 1639	(01)	1.27484 29807 5
2	-5.66882 77802 6550	(02)	4.40258 85815 9
3	-4.16714 64701 7489	(03)	4.57583 83069 4
4	-2.14678 07436 4341	(03)	1.08207 88328 1

TABLE VIIIC - CONTINUED

n = 5

s	p_s		q_s
0	-1.53989 73380 19247	(-01)	1.00000 00000 0000
1	-2.95907 82318 55258	(01)	1.98543 98135 4444
2	-1.66186 70871 83632	(03)	1.19669 31345 0800
3	-3.11279 64549 20657	(04)	2.62557 38645 1274
4	-1.57353 62868 19766	(05)	1.96905 58293 1111
5	-5.57488 41137 46041	(04)	3.62908 13333 1331

n = 6

s	p_s		q_s
0	-1.53989 73381 97693 16	(-01)	1.00000 00000 0000
1	-4.31710 15782 33575 68	(01)	2.86733 19497 5899
2	-3.87754 14174 63784 93	(03)	2.69183 18039 6242
3	-1.35678 86781 37563 47	(05)	1.02878 69305 6687
4	-1.77758 95083 80296 76	(06)	1.62095 60050 0231
5	-6.66907 06166 86364 16	(06)	9.38695 86253 1635
6	-1.72590 22465 48368 45	(06)	1.40622 44112 3580

TABLE IX

$$|\delta C(x)| \leq B|\delta f(x)| + C|\delta g(x)|$$

$$|\delta S(x)| \leq D|\delta f(x)| + E|\delta g(x)|$$

Interval	B	C	D
$1.6 \leq \lvert X\rvert \leq 1.9$	.625	.069	.541
$1.9 \leq \lvert X\rvert \leq 2.4$	.436	.039	.500
$2.4 \leq \lvert X\rvert$	.369	.021	.369

NUMERICAL INTEGRATION OVER A SPHERE

CHRISTOPHER A. FEUCHTER

See article in this issue for explanation of symbols in tab

TABLE 1

The roots r_k and weights C_k of $G_{m+1}(3/2, 3/2, r^2)$, m=0(1)

m = 0

0.33333333333333333333	0.77459666924

m = 1

0.13877799911553081507	0.53846931010
0.19455533421780251827	0.90617984593

m = 2

0.06289133716441942398	0.40584515137
0.15380118384095636775	0.74153118559
0.11664081232795754160	0.94910791234

m = 3

0.03284025994586209607	0.32425342340
0.09804813271549816746	0.61337143270
0.12626367286460207059	0.83603110732
0.07618126780737099922	0.96816023950

m = 4

0.01909367337020706716	0.26954315595
0.06283657634659116753	0.51909612920
0.09931540074741397873	0.73015200557
0.09881668814540756267	0.88706259976
0.05327099472371355724	0.97822865814

m = 5

0.01201813399575544179	0.23045831595