Concerning Two Series for the Gamma Function

By John W. Wrench, Jr.

1. Introduction. It does not seem to be widely recognized that the Stirling asymptotic series for $\Gamma(x)$ yields accurate values for small integer arguments. However, Salzer [1] has pointed out the effectiveness of this series in approximating $\Gamma(z)$ for large values of $|z|$, even when $R(z)$ is quite small. Although the Stirling series for $\ln \Gamma(z)$ contains only odd powers of z^{-1}, whereas the corresponding series for $\Gamma(z)$ contains all powers of z^{-1}, nevertheless the latter provides an effective computational tool for the direct evaluation of $\Gamma(z)$, especially by means of modern digital computers.

For that reason, the exact (rational) values of the first twenty coefficients of Stirling's asymptotic series for $\Gamma(z)$ have been calculated and are tabulated herein.

The second series here considered is the power series for the entire function $1/\Gamma(z)$. The first extensive calculation of the coefficients of this series appears to have been performed by Bourguet [2]. His 16D approximations were subsequently recalculated and corrected by Isaacson and Salzer [3]. These emended values have been reproduced in Davis [4] and in the NBS *Handbook* [5]. In the course of checking [6] these corrected values the present author has now recalculated these coefficients anew and extended the approximations to 31D. These new data are also tabulated in this paper, and their application is illustrated through the evaluation of the main minimum of $\Gamma(x)$ to 31D.

2. Stirling's Asymptotic Series for $\Gamma(z)$. The coefficients of the Stirling series for $\Gamma(z)$ can be derived as follows. Let

$$\Gamma(z) = (2\pi/z)^{1/2} z^z e^{-z} G(z) . \tag{1}$$

Then logarithmic differentiation yields

$$\Gamma'(z)/\Gamma(z) = -1/(2z) + \ln z + G'(z)/G(z) . \tag{2}$$

Next, we apply logarithmic differentiation to the Stirling series for $\ln \Gamma(z)$; namely,

$$\ln \Gamma(z) \sim (z - \tfrac{1}{2}) \ln z - z + \tfrac{1}{2} \ln (2\pi) + \frac{B_2}{1\cdot 2z} + \frac{B_4}{3\cdot 4z^3} + \frac{B_6}{5\cdot 6z^5} + \cdots , \tag{3}$$

where $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $\cdots$ are the Bernoulli numbers. We thereby obtain the following well-known series for the psi function, which also can be obtained directly by means of Watson's lemma [7]:

$$\psi(z) = \Gamma'(z)/\Gamma(z) \sim -\frac{1}{2z} + \ln z - \frac{B_2}{2z^2} - \frac{B_4}{4z^4} - \frac{B_6}{6z^6} - \cdots . \tag{4}$$

Comparing this expansion with that in (2), we infer that

Received December 20, 1967.

$$G'(z)/G(z) \sim -\frac{B_2}{2z^2} - \frac{B_4}{4z^4} - \frac{B_6}{6z^6} - \cdots . \tag{5}$$

We next assume that

$$G(z) \sim 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \tag{6}$$

and that $G'(z)$ also possesses an asymptotic expansion; then

$$G'(z) \sim -\frac{c_1}{z^2} - \frac{2c_2}{z^3} - \frac{3c_3}{z^4} - \cdots . \tag{7}$$

By Eq. (5) we can then write

$$G'(z) \sim \left(-\frac{B_2}{2z^2} - \frac{B_4}{4z^4} - \frac{B_6}{6z^6} - \cdots\right)\left(1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots\right). \tag{8}$$

Expanding the product and comparing coefficients of like powers of z^{-1} in (7), we obtain the recurrence formulas

$$(2k-1)c_{2k-1} = \frac{B_2}{2}c_{2k-2} + \frac{B_4}{4}c_{2k-4} + \cdots + \frac{B_{2k}}{2k}, \tag{9}$$

$$2kc_{2k} = \frac{B_2}{2}c_{2k-1} + \frac{B_4}{4}c_{2k-3} + \cdots + \frac{B_{2k}}{2k}c_1, \tag{10}$$

where $k = 1, 2, 3, \cdots$, and $c_0 = 1$.

Since

$$B_{2n} = (-1)^{n+1}(2n)!\,2^{-2n+1}\pi^{-2n}\zeta(2n), \tag{11}$$

we infer that $|B_{2n}/B_{2n-2}| \sim n(n - \frac{1}{2})/\pi^2$, and therefore

$$c_{2k-1} \sim B_{2k}/2k(2k-1), \tag{12}$$

$$c_{2k} \sim B_{2k}c_1/(2k)^2 = B_{2k}/(48k^2). \tag{13}$$

Consequently, for large k, $c_{2k} \approx c_{2k-1}/12$. Furthermore, we observe from (3) that $B_{2k}/2k(2k-1)$ is simply the coefficient of z^{-2k+1} in Stirling's series for $\ln \Gamma(z)$. It is interesting to note that the decimal values of c_{2k-1} agree to at least two or three significant figures with those of $B_{2k}/2k(2k-1)$ for $k = 1(1)15$. This comparison was facilitated by the extensive decimal table of the latter coefficients calculated by Uhler [8].

F. D. Murnaghan and this writer [9] have derived asymptotic series for the coefficients c_i, of which the leading terms are

$$c_{2k+1} \sim (-1)^k\left[1 - \frac{3}{2(4k-1)} + \cdots\right]\psi_{2k+1}, \tag{14}$$

$$c_{2k+2} \sim (-1)^k\left[\frac{1}{3(4k+1)} - \frac{5}{6(4k+1)(4k-1)} + \cdots\right]\pi\psi_{2k+2}, \tag{15}$$

where

$$\psi_j = \frac{(2j-3)!!(2j+1)!!}{2^{2j}\pi^j(2j)!!}.$$

TABLE 1

The first twenty coefficients in the Stirling asymptotic series for $\Gamma(z)$

c_0	1
c_1	$\frac{1}{12}$
c_2	$\frac{1}{288}$
c_3	$-\frac{139}{51840}$
c_4	$-\frac{571}{2488320}$
c_5	$\frac{163, 879}{209, 018, 880}$
c_6	$\frac{5, 246, 819}{75, 246, 796, 800}$
c_7	$-\frac{534, 703, 531}{902, 961, 561, 600}$
c_8	$-\frac{4, 483, 131, 259}{86, 684, 309, 913, 600}$
c_9	$\frac{432, 261, 921, 612, 371}{514, 904, 800, 886, 784, 000}$
c_{10}	$\frac{6, 232, 523, 202, 521, 089}{86, 504, 006, 548, 979, 712, 000}$
c_{11}	$-\frac{25, 834, 629, 665, 134, 204, 969}{13, 494, 625, 021, 640, 835, 072, 000}$
c_{12}	$-\frac{1, 579, 029, 138, 854, 919, 086, 429}{9, 716, 130, 015, 581, 401, 251, 840, 000}$
c_{13}	$\frac{746, 590, 869, 962, 651, 602, 203, 151}{116, 593, 560, 186, 976, 815, 022, 080, 000}$
c_{14}	$\frac{1, 511, 513, 601, 028, 097, 903, 631, 961}{2, 798, 245, 444, 487, 443, 560, 529, 920, 000}$
c_{15}	$-\frac{8, 849, 272, 268, 392, 873, 147, 705, 987, 190, 261}{299, 692, 087, 104, 605, 205, 332, 754, 432, 000, 000}$
c_{16}	$-\frac{142, 801, 712, 490, 607, 530, 608, 130, 701, 097, 701}{57, 540, 880, 724, 084, 199, 423, 888, 850, 944, 000, 000}$
c_{17}	$\frac{2, 355, 444, 393, 109, 967, 510, 921, 431, 436, 000, 087, 153}{13, 119, 320, 805, 091, 197, 468, 646, 658, 015, 232, 000, 000}$
c_{18}	$\frac{2, 346, 608, 607, 351, 903, 737, 647, 919, 577, 082, 115, 121, 863}{155, 857, 531, 164, 483, 425, 927, 522, 297, 220, 956, 160, 000, 000}$
c_{19}	$-\frac{2, 603, 072, 187, 220, 373, 277, 150, 999, 431, 416, 562, 396, 331, 667}{1, 870, 290, 373, 973, 801, 111, 130, 267, 566, 651, 473, 920, 000, 000}$
c_{20}	$-\frac{73, 239, 727, 426, 811, 935, 976, 967, 471, 475, 430, 268, 695, 630, 993}{628, 417, 565, 655, 197, 173, 339, 769, 902, 394, 895, 237, 120, 000, 000}$

TABLE 2

50D values of the first twenty coefficients in the asymptotic series for $\Gamma(z)$

c_1	0.08$\dot{3}$									
c_2	0.00347	$\dot{2}$								
c_3	−0.00268	13$\dot{2}$71	60493	$\dot{8}$						
c_4	−0.00022	94720	$\dot{9}$3621	39917	69547	32510	28806	58$\dot{4}$		
c_5	0.00078	40392	21720	06662	74740	34881	44228	88496	96257	10366
c_6	0.00006	97281	37583	65857	77429	39882	85757	83308	29359	63594
c_7	−0.00059	21664	37353	69388	28648	36225	60440	11873	91585	19680
c_8	−0.00005	17179	09082	60592	19337	05784	30020	58822	81785	34534
c_9	0.00083	94987	20672	08727	99933	57516	76498	34451	98182	11159
c_{10}	0.00007	20489	54160	20010	55908	57193	02250	15052	06345	17380
c_{11}	−0.00191	44384	98565	47752	65008	98858	32852	25448	76893	57895
c_{12}	−0.00016	25162	62783	91581	68986	35123	98027	09981	05872	59193
c_{13}	0.00640	33628	33808	06979	48236	38090	26579	58304	01893	93280
c_{14}	0.00054	01647	67892	60451	51804	67508	57024	17355	47254	41598
c_{15}	−0.02952	78809	45699	12050	54406	51054	69382	44465	65482	82544
c_{16}	−0.00248	17436	00264	99773	09156	58368	74346	43239	75168	04723
c_{17}	0.17954	01170	61234	85610	76994	07722	22633	05309	12823	38692
c_{18}	0.01505	61130	40026	42441	23842	21877	13112	72602	59815	45541
c_{19}	−1.39180	10932	65337	48139	91477	63542	27314	93580	45617	72646
c_{20}	−0.11654	62765	99463	20085	07340	36907	14796	96789	37334	38371

Note. The first four entries are shown as repeating decimals, the repetends being indicated by superior dots.

We infer from these relations that

$$c_{2k}/c_{2k-1} \sim \frac{1}{12}\left(1 + \frac{1}{4k+1}\right), \tag{16}$$

which refines our previous estimate of the relative magnitudes of c_{2k} and c_{2k-1}.

By means of Eqs. (9) and (10), the tabulated exact values of the coefficients c_i were successively calculated for $i = 1(1)20$. The accuracy of the first nine coefficients as published by Davis [10] is confirmed.

For ease in application to specific calculations, a table of 50D equivalents of these exact coefficients is also included. (The first four entries are shown as repeating decimals in this range.) Several entries beyond the range of this table were evaluated [9] to 6 or 7S by the asymptotic series (14) and (15). For completeness, these supplementary values are reproduced here in Table 3.

TABLE 3

Supplementary values of the coefficients in the asymptotic series for $\Gamma(z)$

j	c_j
21	13.3980
22	1.12093
23	−156.802
24	−13.1088
25	2192.56
26	183.199
27	−36101.1
28	−3015.17
29	691346.4
30	57722.53

3. Power Series for $1/\Gamma(z)$. We start with Legendre's series

$$(17)\qquad \ln \Gamma(1-z) = \gamma z + \sum_{k=2}^{\infty} S_k z^k/k, \qquad |z| < 1,$$

where γ is Euler's constant and $S_k = \zeta(k) = \sum_{k=1}^{\infty} n^{-k}$. This series in combination with the reflection formula

$$(18)\qquad \Gamma(z)\Gamma(1-z) = \pi/\sin \pi z, \qquad 0 < R(z) < 1$$

and the series

$$(19)\qquad \ln\left(\frac{\sin \pi z}{\pi z}\right) = -\sum_{k=1}^{\infty} S_{2k} z^{2k}/k$$

yields the series

$$(20)\qquad \ln [z\Gamma(z)] = -\gamma z + \sum_{k=2}^{\infty} (-1)^k S_k z^k/k, \qquad |z| < 1,$$

from which we infer

$$(21)\qquad \frac{1}{\Gamma(z)} = z \exp\left\{\gamma z - \sum_{k=2}^{\infty} (-1)^k S_k z^k/k\right\},$$

whence we obtain

$$(22)\qquad \frac{1}{\Gamma(z)} = \sum_{k=1}^{\infty} a_k z^k,$$

where

$$a_1 = 1, \qquad a_2 = \gamma,$$

and

$$i a_i = \gamma a_{i-1} - S_2 a_{i-2} + S_3 a_{i-3} - \cdots + (-1)^{i+1} S_i \qquad (i \geqq 2).$$

This recurrence formula was originally derived by Bourguet [2] in a different manner, starting with Euler's infinite product for $\Gamma(z)$.

Since

$$(23)\qquad \ln(1+z) = \sum_{k=1}^{\infty} (-1)^{k+1} z^k/k, \qquad |z| < 1,$$

we infer that Eq. (20) is equivalent to

$$(24)\qquad \ln [z(1+z)\Gamma(z)] = (1-\gamma)z + \sum_{k=2}^{\infty} (-1)^k S_k' z^k/k,$$

where $S_k' = \zeta(k) - 1 = \sum_{n=2}^{\infty} n^{-k}$.

Hence, if we write

$$(25)\qquad \frac{1}{\Gamma(z)} = z(1+z)[b_0 + b_1 z + b_2 z^2 + \cdots],$$

we find

$$
\begin{aligned}
&b_0 = 1 , \qquad b_1 = \gamma - 1 , \\
&ib_i = (\gamma - 1)b_{i-1} - S_2'b_{i-2} + S_3'b_{i-3} - \cdots + (-1)^{i+1}S_i' \qquad (i \geqq 2) .
\end{aligned} \tag{26}
$$

Moreover, the coefficients in series (22) and (25) are connected by the relation

$$a_i = b_{i-1} + b_{i-2} , \qquad (i \geqq 2) . \tag{27}$$

It was found more convenient to calculate the coefficients b_i in succession from Eq. (26) and then to deduce the corresponding values of a_i from Eq. (27).

Several omnibus checks have been applied to the tabulated values a_i^*, b_i^* of the coefficients a_i and b_i. These results are

$$\sum_{i=1}^{41} a_i^* = 1 + 4 \cdot 10^{-31} ,$$

$$\sum_{i=0}^{39} b_i^* = \frac{1}{2} + 2 \cdot 10^{-31} ,$$

$$\sum_{i=1}^{34} a_i^*/2^i = \pi^{-1/2} + 2 \cdot 10^{-32} .$$

A further partial check on the accuracy of these approximations to a_i was made possible through the kindness of Yudell L. Luke, who sent the author an unpublished table of these coefficients calculated to about 28D at Midwest Research Institute by Rosemary Moran under his direction. Agreement of these results with those of the author to at least 27D was noted.

TABLE 4

Coefficients b_i to 31D

i	b_i						
1	−0.42278	43350	98467	13939	34879	09917	6
2	−0.23309	37364	21786	74168	35316	05227	8
3	0.19109	11013	87691	50615	45276	70352	4
4	−.02455	24900	05400	01665	28268	75250	3
5	−.01764	52445	50144	32009	53814	26038	9
6	$.0802$	32730	22267	34653	32665	04366	6
7	$-.0_{2}80$	43297	75604	24699	08714	94026	1
8	$-.0_{3}36$	08378	16254	81812	12424	77057	9
9	$.0_{3}14$	55961	42139	86714	84267	47094	8
10	$-.0_{3}1$	75458	59751	75096	22735	48468	5
11	$-.0_{4}$	25889	95029	03727	63821	40922	9

$_{5}$

12	$.0_{5}$	13385	01546	89460	57247	95563	4
13	$-.$	$0_{6}2054$	74314	91290	98424	21434	6
14		$-.01_{9}$	59526	78485	08679	23581	3
15		$.062_{8}$	75621	88933	22837	41444	7
16		$-.012_{8}$	73614	24486	30608	11388	5
17		$.0_{10}$	92339	67437	60406	66800	2
18		$.0_{10}$	12002	99679	30693	84248	6
19		$-.$	$0_{11}4220$	73335	31643	12994	9
20			$.0_{12}523$	92773	45221	07286	7
21			$-.013_{13}$	89070	57766	59688	8
22			$-.06_{14}$	69255	47590	05379	1
23			$.01_{14}$	34443	22195	82361	1
24			$-.0_{15}$	11765	35913	44100	2
25				$-.047_{18}$	23388	25645	4
26				$.0165_{17}$	90310	80397	1
27				$-.024_{18}$	66504	25079	1
28				$.01_{19}$	67758	56635	5
29				$.$	$0_{21}3682$	06583	8
30				$-.$	$0_{21}2344$	71410	7
31					$.0_{22}290$	48055	6
32					$-.016_{23}$	87755	0
33					$-.0_{25}$	44601	4
34					$.0_{25}$	20995	4
35					$-.$	$0_{26}2345$	4
36						$.0_{27}127$	4
37						$.02_{29}$	5
38						$-.01_{29}$	3
39						$.0_{30}$	1

4. The Main Minimum of $\Gamma(x)$. As an example of a nontrivial application of Table 5, the main minimum of the factorial function $x!$, or $\Gamma(1 + x)$, has been evaluated thereby to 31D.

The abscissa of this minimum was calculated from the equation

$$\begin{aligned}\psi(x) &= \frac{\Gamma'(1+x)}{\Gamma(1+x)} \\ &= \frac{1}{2x} - \frac{1}{1-x^2} - \frac{\pi}{2}\cot \pi x + (1-\gamma) - S_3'x^2 - S_5'x^4 - \cdots = 0\ ,\end{aligned} \tag{28}$$

starting with a 15D approximation due to J. C. P. Miller [11] and applying Newton-Raphson iteration. The required abscissa to 33D was thus found to be

$$x_0 = 0.46163\ \ 21449\ \ 68362\ \ 34126\ \ 26595\ \ 42325\ \ 721\ \ \cdots .$$

Then $1/\Gamma(1 + x_0)$, $= 1/x_0\Gamma(x_0)$, was evaluated from series (22), using synthetic division and the coefficients in Table 5. This calculation yielded the approximation

$$1/\Gamma(1 + x_0) = 1.12917\ \ 38854\ \ 50141\ \ 23991\ \ 36073\ \ 09471\ \ 1\cdots ,$$

whose reciprocal is

$$\Gamma(1 + x_0) = 0.88560\ \ 31944\ \ 10888\ \ 70027\ \ 88159\ \ 00582\ \ 6\cdots .$$

This confirms the accuracy of the 15D approximation given by Miller.

TABLE 5

Coefficients a_i to 31D

i	a_i						
2	0.57721	56649	01532	86060	65120	90082	4
3	−0.65587	80715	20253	88107	70195	15145	4
4	−.04200	26350	34095	23552	90039	34875	4
5	.16653	86113	82291	48950	17007	95102	1
6	−.04219	77345	55544	33674	82083	01289	2
7	$-.0_2962$	19715	27876	97356	21149	21672	3
8	$.0_2721$	89432	46663	09954	23950	10340	5
9	$-.0_2116$	51675	91859	06511	21139	71084	0
10	$-.0_321$	52416	74114	95097	28157	29963	1
11	$.0_312$	80502	82388	11618	61531	98626	3
12	$-.0_42$	01348	54780	78823	86556	89391	4
13	$-.0_5$	12504	93482	14267	06573	45359	5

14	$.0_{5}$ 11330 27231 98169 58823 74128 9
15	$-.0_{6}2056$ 33841 69776 07103 45015 9
16	$.0_{8}61$ 16095 10448 14158 17863 4
17	$.0_{8}50$ 02007 64446 92229 30056 2
18	$-.0_{8}11$ 81274 57048 70201 44588 3
19	$.0_{9}1$ 04342 67116 91100 51048 8
20	$.0_{11}7782$ 26343 99050 71253 7
21	$-.0_{11}3696$ 80561 86422 05708 2
22	$.0_{12}510$ 03702 87454 47597 9
23	$-.0_{13}20$ 58326 05356 65067 9
24	$-.0_{14}5$ 34812 25394 23018 0
25	$.0_{14}1$ 22677 86282 38260 9
26	$-.0_{15}$ 11812 59301 69745 6
27	$.0_{17}118$ 66922 54751 7
28	$.0_{17}141$ 23806 55318 0
29	$-.0_{18}22$ 98745 68443 6
30	$.0_{19}1$ 71440 63219 3
31	$.0_{21}1337$ 35173 1
32	$-.0_{21}2054$ 23355 1
33	$.0_{22}273$ 60300 6
34	$-.0_{23}17$ 32356 4
35	$-.0_{25}$ 23606 0
36	$.0_{25}$ 18650 0
37	$-.0_{26}2218$ 0
38	$.0_{27}129$ 9
39	$.0_{29}1$ 2
40	$-.0_{29}1$ 1
41	$.0_{30}$ 1

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1. *Table of the Gamma Function for Complex Arguments*, Nat. Bur. Standards Appl. Math. Series, 34, U. S. Government Printing Office, Washington, D. C., 1954, p. viii. MR **16**, 403.

2. L. Bourguet, "Sur les intégrales Euleriennes et quelques autres fonctions uniformes," *Acta Math.*, v. 2, 1883, pp. 261–295.

3. *MTAC*, v. 1, 1943, p. 124, MTE **19**.

4. H. T. Davis, *Tables of the Mathematical Functions*, Vol. I, Trinity Univ. Press, Texas, 1963, p. 185.

5. Milton Abramowitz & I. A. Stegun, (Editors), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Nat. Bur. Standards Appl. Math. Series, 55, Superintendent of Documents, U. S. Government Printing Office, Washington, D. C., 1964, p. 256; 3rd printing, with corrections, 1965. MR **29** #4914; MR **31** #1400.

6. *Math. Comp.*, v. 20, 1966, p. 641, MTE **399**. (The final three digits in the 28D value of c_{23} given in the editorial footnote should read 650 instead of 479.)

7. E. T. Copson, *Asymptotic Expansions*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 55, Cambridge Univ. Press, New York, 1964, pp. 51–52. MR **29** #6234.

H. S. Uhler, "The coefficients of Stirling's series for log $\Gamma(x)$," *Proc. Nat. Acad. Sci. U.S.A.*, v. 28, 1942, pp. 59–62. MR **3**, 275.

9. F. D. Murnaghan & J. W. Wrench, Jr., *The Converging Factor for the Exponential Integral*, David Taylor Model Basin Report 1535, 1963.

10. H. T. Davis, *loc. cit.*, pp. 180–181.

11. A. Fletcher, J. C. P. Miller, L. Rosenhead & L. J. Comrie, *An Index of Mathematical Tables*, Vol. I, 2nd ed., Addison-Wesley, Reading, Mass., 1962, p. 295. MR **26** #365a.