Rational Chebyshev Approximations for the Exponential Integral $E_1(x)$

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Abstract. Rational Chebyshev approximations are presented for the exponential integral $E_1(x)$ in the intervals (0, 1], [1, 4], and [4, ∞) with maximal relative errors ranging down to 10^{-21}. 25S coefficients are also given for a continued-fraction expansion for small x.

1. Introduction. Accurate function values for the exponential integral $E_1(x)$, defined [1] by

$$(1) \qquad E_1(x) = -\text{Ei}(-x) = \int_x^\infty \frac{e^{-t}}{t}\,dt\,,$$

are frequently needed in a variety of physical applications. Approximations to this function previously given by Allen [2] and Hastings [3], [4] are of insufficient accuracy for many such applications and for the longer word-length of modern computers. While Clenshaw [5] has given 20D coefficients for expansions in Chebyshev polynomials, such expansions are generally less efficient than rational forms. In addition, the Clenshaw form loses significance through subtraction for $1 < x \leqq 4$. In this paper we present a set of nearly-best rational approximations for $E_1(x)$ for $0 < x < \infty$ with accuracies up to 20S. The approximation forms used are efficient and are computationally stable.

2. Approximation Forms. The approximation forms and corresponding intervals used are

$$(2) \qquad \begin{aligned} E_{lm}(x) &= -\ln(x) + R_{lm}(x)\,, \qquad 0 < x \leqq 1\,,\\ &= e^{-x}R_{lm}(1/x)\,, \qquad 1 \leqq x \leqq 4\,,\\ &= \frac{e^{-x}}{x}\left[1 + \frac{1}{x}R_{lm}(1/x)\right], \qquad 4 \leqq x < \infty\,, \end{aligned}$$

where the $R_{lm}(z)$ are rational functions of degree l in the numerator and m in the denominator. All of the previously mentioned approximations have made use of the first and last of these forms, but not necessarily over the same intervals we use. The first form, based on the analytic expansion

$$(3) \qquad E_1(x) + \ln(x) = -\gamma - \sum_{n=1}^{\infty} \frac{(-x)^n}{n\cdot n!}\,,$$

where γ is Euler's constant, involves a loss of significance through subtraction of

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nearly equal quantities when used for $x > 1$. Thus, for example, Clenshaw loses up to three significant figures by using this form for $|x| \leqq 4$. The third form, based on the continued-fraction expansion of $E_1(x)$ for large x, is not efficient when used for the entire interval $[1, \infty)$. The combination of the second form and interval given above is the best of many such combinations tried.

3. Computations. Final computations were carried out in 25S arithmetic on a CDC 3600. The approximations were computed without difficulty using standard versions of the Remes algorithm for rational Chebyshev approximation [6], [7]. Function values were computed as needed using one of two techniques, depending on the value of the argument. The expansion (3) converges for all x, but for x appreciably greater than 1, the rate of convergence is slow, and individual terms become large, causing subtraction errors. This series was transformed by means of

TABLE I

$$\mathcal{E}_{lm} = -100 \log_{10} \max \left| \frac{E_1(x) - E_{lm}(x)}{E_1(x)} \right|$$

$m \backslash l$	0	1	2	3	4	5	6	7	8	9
					$0 < x \leqq 1$					
0	33	156	282	415	555	701	852	1008	1169	
1		394	492	633	783	940	1101	1266		
2			607	818	988	1157	1328	1502		
3				1086	1195	1367	1546			
4					1310	1565				
5						1854				
6							2095			
					$1 \leqq x \leqq 4$					
0	31	145	242	330	412	491	567	641	713	783
1		286	401	503	597	685	769			
2		386	531	645	749	846	937			
3		473	637	769	882	987	1086			
4		554	732	877	1004	1116				
5		631				1235				
6							1465			
7								1693		
8									1920	
					$4 \leqq x$					
0	193	293	382	464	541	614	683	750	814	
1		458	552	639	720	797				
2		591	691	782	868	950				
3		687	812	909	998					
4		772	906	1022	1116					
5						1317				
6							1512			
7								1702		
8									1888	
9										2071

TABLE II

$$E_1(x) \simeq -\ln(x) + \sum_{s=0}^{n} p_s x^s \Big/ \sum_{s=0}^{n} q_s x^s, \qquad 0 < x \leqq 1$$

n	s	p_s	q_s
1	0	−2.22420 7 (00)	3.85031 1 (00)
	1	3.28841 0 (00)	1.00000 0 (00)
2	0	−4.43668 255 (00)	7.68641 124 (00)
	1	4.42054 938 (00)	5.65655 216 (00)
	2	3.16274 620 (00)	1.00000 000 (00)
3	0	−1.64330 27495 86 (02)	2.84694 75956 20 (02)
	1	2.21255 49175 58 (02)	1.09905 66056 19 (02)
	2	2.91730 56556 03 (01)	1.65603 61842 05 (01)
	3	4.32318 05992 45 (00)	1.00000 00000 00 (00)
4	0	−4.34981 43832 95212 0 (02)	7.53585 64359 84293 2 (02)
	1	4.25696 82638 59170 3 (02)	5.68052 52718 98695 5 (02)
	2	2.92525 18866 92054 9 (02)	1.50950 38744 25130 9 (02)
	3	3.98941 53870 32106 6 (01)	1.88992 88395 00296 9 (01)
	4	4.29312 52343 20972 6 (00)	1.00000 00000 00000 0 (00)
5	0	−4.41785 47172 82165 45601 (04)	7.65373 32333 76136 20990 (04)
	1	5.77217 24713 94443 45017 (04)	3.25971 88129 02747 27384 (04)
	2	9.93831 38896 20368 10877 (03)	6.10610 79424 57594 97347 (03)
	3	1.84211 08866 79996 39377 (03)	6.35419 41837 83817 02242 (02)
	4	1.01093 80616 19055 33427 (02)	3.72298 35283 33274 27940 (01)
	5	5.03416 18409 75683 92766 (00)	1.00000 00000 00000 00000 (00)
6	0	−1.48151 02102 57575 08380 86 (05)	2.56664 93484 89711 73192 68 (05)
	1	1.50260 59476 43698 24207 37 (05)	1.84340 70063 35367 73592 98 (05)
	2	8.99049 72007 45725 65532 51 (04)	5.24405 29172 05635 54298 83 (04)
	3	1.59241 75980 63730 36398 84 (04)	8.12580 35174 76873 57598 66 (03)
	4	2.15006 72908 09291 81232 09 (03)	7.50431 63907 10393 66241 65 (02)
	5	1.16695 52669 73446 10833 68 (02)	4.02054 65640 02770 60614 33 (01)
	6	5.01967 85185 43984 37910 20 (00)	1.00000 00000 00000 00000 00 (00)

TABLE III

$$E_1(x) \simeq e^{-x} \sum_{s=0}^{n} p_s x^{-s} \Big/ \sum_{s=0}^{n} q_s x^{-s}, \qquad 1 \leqq x \leqq 4$$

n	s	p_s		q_s	
1	0	1.4867	(−02)	1.0000	(00)
	1	8.6607	(−01)	4.7924	(−01)
2	0	1.21022 05	(−03)	1.00000 00	(00)
	1	9.81479 89	(−01)	1.61986 45	(00)
	2	7.53397 42	(−01)	2.91351 51	(−01)
3	0	1.33017 0319	(−04)	1.00000 0000	(00)
	1	9.97074 3175	(−01)	3.22255 1838	(00)
	2	2.25390 9654	(00)	2.08498 9311	(00)
	3	6.30676 6097	(−01)	2.01741 7317	(−01)
4	0	1.77505 46228 63	(−05)	1.00000 00000 00	(00)
	1	9.99485 00852 88	(−01)	5.19483 84424 40	(00)
	2	4.20224 42231 86	(00)	6.93251 45941 50	(00)
	3	3.66443 11027 66	(00)	2.49230 34830 53	(00)
	4	5.39370 23406 26	(−01)	1.52272 58442 11	(−01)
5	0	2.72428 14258 1277	(−06)	1.00000 00000 0000	(00)
	1	9.99901 07561 7661	(−01)	7.48056 46585 6657	(00)
	2	6.48236 27686 3502	(00)	1.64853 53756 8922	(01)
	3	1.09823 28074 9419	(01)	1.25009 56251 1548	(01)
	4	5.20167 43818 4078	(00)	2.89289 71153 3891	(00)
	5	4.75185 76444 3999	(−01)	1.22430 89892 0495	(−01)

6	0	4.65627 10797 50956 6 (−07)	1.00000 00000 00000 0 (00)
	1	9.99979 57705 15949 7 (−01)	1.00411 64382 90544 8 (01)
	2	9.04161 55694 63286 6 (00)	3.24264 21069 51380 5 (01)
	3	2.43784 08879 13167 3 (01)	4.12807 84189 14243 4 (01)
	4	2.30192 55939 13334 6 (01)	2.04494 78501 37941 7 (01)
	5	6.90522 52278 44435 7 (00)	3.31909 21359 33016 0 (00)
	6	4.30967 83946 93887 8 (−01)	1.03400 13040 48739 8 (−01)
7	0	8.67745 95483 84437 437 (−08)	1.00000 00000 00000 000 (00)
	1	9.99995 51930 13903 006 (−01)	1.28481 93537 91566 499 (01)
	2	1.18483 10555 49458 443 (01)	5.64433 56956 18031 986 (01)
	3	4.55930 64425 33898 233 (01)	1.06645 18376 99138 825 (02)
	4	6.99279 45129 10030 229 (01)	8.97311 09712 52898 022 (01)
	5	4.25202 03476 88407 791 (01)	3.14971 84917 04407 502 (01)
	6	8.83671 80880 38439 386 (00)	3.79559 00376 21222 428 (00)
	7	4.01377 66494 06647 203 (−01)	9.08804 56918 88692 188 (−02)
8	0	1.73733 17607 20576 03093 2 (−08)	1.00000 00000 00000 00000 0 (00)
	1	9.99998 96423 47613 06843 7 (−01)	1.58796 45707 58947 92790 3 (01)
	2	1.48796 77028 40464 06661 3 (01)	9.02165 84505 29372 64231 4 (01)
	3	7.63362 88437 05946 89089 6 (01)	2.34257 35047 17625 15305 3 (02)
	4	1.69810 67637 64238 38270 5 (02)	2.95313 63356 77908 51742 3 (02)
	5	1.70063 29783 11516 12932 8 (02)	1.77572 81867 17289 79967 7 (02)
	6	7.24668 97828 58597 02119 9 (01)	4.66217 96103 56861 75681 2 (01)
	7	1.10732 66277 86831 74380 9 (01)	4.34483 63355 09282 08336 0 (00)
	8	3.82857 31210 22477 16910 8 (−01)	8.25816 00085 64488 03469 8 (−02)

TABLE IV

$$E_1(x) \simeq \frac{e^{-x}}{x}\left[1 + 1/x \sum_{s=0}^{n} p_s x^{-s} \Big/ \sum_{s=0}^{n} q_s x^{-s}\right], \qquad 4 \leqq x$$

n	s	p_s		q_s	
0	0	−7.370	(−01)	1.000	(00)
1	0	−9.96969 8	(−01)	1.00000 0	(00)
	1	−4.25784 9	(−01)	2.31826 1	(00)
2	0	−9.99964 218	(−01)	1.00000 000	(00)
	1	−3.73313 946	(00)	5.72998 011	(00)
	2	−4.46320 868	(−01)	6.00442 247	(00)
3	0	−9.99999 54096 2	(−01)	1.00000 00000 0	(00)
	1	−8.13255 72741 8	(00)	1.01324 79271 3	(01)
	2	−1.24514 81283 9	(01)	2.67210 71019 7	(01)
	3	−6.42505 77883 8	(−01)	1.71463 13845 1	(01)
4	0	−9.99999 99359 628	(−01)	1.00000 00000 000	(00)
	1	−1.34511 14977 055	(01)	1.54511 13160 377	(01)
	2	−4.79714 42385 415	(01)	7.28738 48194 513	(01)
	3	−4.20689 28555 297	(01)	1.19100 78997 159	(02)
	4	−1.14468 73348 374	(00)	5.32225 12691 417	(01)
5	0	−9.99999 99990 36009	(−01)	1.00000 00000 00000	(00)
	1	−1.96304 08535 93865	(01)	2.16304 08494 23774	(01)
	2	−1.19557 61038 37175	(02)	1.56818 43364 53856	(02)
	3	−2.54376 33976 88951	(02)	4.62230 27156 14783	(02)
	4	−1.47982 19500 50448	(02)	5.30685 09610 81167	(02)
	5	−2.39099 64453 13573	(00)	1.77600 70940 35063	(02)

6	0	−9.99999 99999 84469 10	(−01)	1.00000 00000 00000 00	(00)
	1	−2.66271 06043 18114 52	(01)	2.86271 06042 21918 96	(01)
	2	−2.41055 82709 70148 47	(02)	2.92310 03938 85332 47	(02)
	3	−8.95927 95777 29368 14	(02)	1.33278 53774 82572 31	(03)
	4	−1.29885 68874 64841 01	(03)	2.77761 94950 91629 62	(03)
	5	−5.45374 15888 31328 73	(02)	2.40401 71322 59089 47	(03)
	6	−5.66575 20653 38687 35	(00)	6.31657 48328 08002 29	(02)
7	0	−9.99999 99999 99734 1443	(−01)	1.00000 00000 00000 0000	(00)
	1	−3.44061 99500 66848 9491	(01)	3.64061 99500 64598 0400	(01)
	2	−4.27532 67120 19885 3941	(02)	4.94345 07020 99036 4527	(02)
	3	−2.39601 94324 74905 4028	(03)	3.19027 23748 95433 0383	(03)
	4	−6.16885 21005 54763 5088	(03)	1.03370 75308 58409 7698	(04)
	5	−6.57609 69874 80211 7925	(03)	1.63241 45355 77835 0289	(04)
	6	−2.10607 73714 26332 8896	(03)	1.11497 75287 10966 2000	(04)
	7	−1.48990 84997 29481 6902	(01)	2.37813 89910 21602 2120	(03)
8	0	−9.99999 99999 99995 19501	(−01)	1.00000 00000 00000 00000	(00)
	1	−4.29386 24114 77102 57520	(01)	4.49386 24114 77048 90829	(01)
	2	−6.95170 14067 13531 07041	(02)	7.79047 38890 11024 39770	(02)
	3	−5.40015 52494 29216 58167	(03)	6.71261 82825 01531 33866	(03)
	4	−2.12023 11271 04813 65124	(04)	3.09117 90486 38225 40045	(04)
	5	−4.06389 28671 55509 35195	(04)	7.62113 01980 23102 09672	(04)
	6	−3.34895 49619 12054 50827	(04)	9.53743 96753 45534 18034	(04)
	7	−8.50640 61297 99729 23790	(03)	5.30984 66398 43201 92097	(04)
	8	−4.28026 86155 59885 31046	(01)	9.42559 71270 68271 36132	(03)
9	0	−9.99999 99999 99999 90878 19	(−01)	1.00000 00000 00000 00000 00	(00)
	1	−5.21996 32588 52257 24810 39	(01)	5.41996 32588 52255 94149 24	(01)
	2	−1.06117 77263 55033 17668 71	(03)	1.16357 69915 32084 80354 59	(03)
	3	−1.08168 52399 09591 56224 98	(04)	1.28428 08586 62729 73659 98	(04)
	4	−5.93468 41538 83711 91723 56	(04)	7.92317 87945 27904 36987 18	(04)
	5	−1.75032 73087 49708 13147 08	(05)	2.78581 34710 52084 21393 57	(05)
	6	−2.61814 54937 20563 96473 81	(05)	5.46168 42050 69115 57357 58	(05)
	7	−1.72833 75773 77759 39268 28	(05)	5.59037 56210 02286 40033 80	(05)
	8	−3.58461 98743 99690 43086 95	(04)	2.59897 62083 60848 97774 11	(05)
	9	−1.32768 81505 63744 46229 87	(02)	3.91478 56245 55634 56270 78	(04)

TABLE V

$$E_1(x) = -\ln(x) - \gamma + \frac{a_1 x}{1-} \frac{a_2 x}{1-} \cdots$$

i	a_i					
1	1.00000	00000	00000	00000	0000	(00)
2	−2.50000	00000	00000	00000	0000	(−01)
3	2.77777	77777	77777	77777	7778	(−02)
4	−2.77777	77777	77777	77777	7778	(−01)
5	1.64000	00000	00000	00000	0000	(−01)
6	−4.81463	41463	41463	41463	4146	(−02)
7	1.86603	29429	42170	56679	5923	(−02)
8	−1.60205	68757	07501	54525	5703	(−01)
9	1.23861	39210	41923	32991	7815	(−01)
10	−2.31794	05112	45808	29915	6950	(−02)
11	1.29681	11650	66612	32574	9017	−(02)
12	−1.12427	07106	81348	55498	0946	(−01)
13	9.48210	68248	17627	18042	2609	(−02)
14	−1.50705	56940	70993	41586	7282	(−02)
15	9.91703	79965	35417	38979	0346	(−03)
16	−8.65217	39735	62029	78930	2325	(−02)
17	7.61891	65113	71489	38461	1831	(−02)
18	−1.11260	95393	81208	17324	8394	(−02)
19	8.02137	38225	66099	94103	6526	(−03)
20	−7.03079	19046	68019	59276	8798	(−02)
21	6.35254	92288	39041	18840	0415	(−02)
22	−8.80574	51173	87599	83602	9885	(−03)
23	6.73156	43159	35130	57355	3415	(−03)
24	−5.92093	68551	88450	02360	8312	(−02)
25	5.44193	72539	14569	37403	6161	(−02)
26	−7.28137	18497	92553	59228	5206	(−03)
27	5.79799	42107	94125	58486	8766	(−03)
28	−5.11362	97085	60959	55012	6918	(−02)
29	4.75746	03552	29856	08437	2345	(−02)
30	−6.20470	20824	22709	71921	3328	(−03)
31	5.09130	38779	10156	55165	2889	(−03)
32	−4.50002	85546	37539	73594	7098	(−02)
33	4.22487	22218	42761	47612	1945	(−02)
34	−5.40431	89612	94834	49318	0440	(−03)
35	4.53788	95713	23179	68530	3408	(−03)
36	−4.01789	51290	74937	23947	5927	(−02)
37	3.79896	24606	33396	75543	1572	(−02)
38	−4.78623	11965	44072	42723	6975	(−03)
39	4.09283	20621	35569	26450	8815	(−03)
40	−3.62907	03638	13851	03882	9943	(−02)

the QD algorithm [8], using 40S arithmetic, into a continued fraction

$$E_1(x) + \ln(x) = -\gamma + \frac{a_1 x}{1-}\,\frac{a_2 x}{1-}\,\frac{a_3 x}{1-}\cdots \tag{4}$$

which converges rapidly with little inherent error except near the zero at $x \simeq .675$. Using this continued fraction for $0 < x \leqq 4$, and the standard continued fraction [1, Eq. 5.1–22] for $x > 4$, at least 23S master function values were obtained for E_1 for all $x > 0$.

4. Results. Table I lists the values of

$$\varepsilon_{lm} = -100 \log_{10} \max \left| \frac{E_1(x) - E_{lm}(x)}{E_1(x)} \right|,$$

where the maximum is taken over the appropriate interval, for the initial segments of the various L_∞ Walsh arrays. Tables II–IV present the coefficients for the main diagonals of these arrays. All coefficients are given to accuracies greater than that justified by the maximal errors, but reasonable additional rounding should not greatly affect the overall accuracies. Each approximation listed, with the coefficients rounded as they appear in the table, was tested against the master function routines for 5000 pseudo-random arguments. In all cases maximal errors agreed, within roundoff, in magnitude and location with those for the unrounded approximations.

Table V lists the coefficients of the continued fraction (4) to 25S. While these coefficients are not all correct to 25S, the evaluation of (4) using these coefficients appears correct to nearly 25S for $x \leqq 8$, (25D for $x \simeq .675$), slowly deteriorating in accuracy for larger x.

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