Error Estimates for the Clenshaw-Curtis Quadrature

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1. Introduction. Clenshaw and Curtis [1] have proposed a quadrature scheme based on the "practical" abscissas $x_i = \cos(i\pi/n)$, $i = 0(1)n$ and they have also discussed the estimation of error of the quadrature formula. Elliott [2] has discussed the estimation of truncation errors in the two Chebyshev series approximations for a function, one based on the practical abscissas and the other on the "classical" abscissas $x_i = \cos(2i + 1)\pi/(2n + 2)$, $i = 0(1)n$. Elliott also obtains asymptotic error estimates for the Lagrangian quadrature formulas based on these two sets of points. Recently, Fraser and Wilson [3] have discussed the estimation of error of the Clenshaw-Curtis quadrature and they give a simple formula for the calculation of the error in terms of the function-values.

In the present note we obtain error estimates for the Clenshaw-Curtis quadrature applied to functions analytic on the interval of integration $[-1, 1]$. We also obtain error estimates for the quadrature formula based on the classical abscissas.

2. The Clenshaw-Curtis Quadrature Formula. Let $\Psi_n(x)$ denote the Lagrangian interpolation polynomial for $f(x)$ at the practical abscissas $x_i = \cos(i\pi/n)$, $i = 0(1)n$, and let $\psi_n(x)$ denote the error of interpolation

$$\psi_n(x) = f(x) - \Psi_n(x)\,. \tag{1}$$

If

$$\Psi_n(x) = \sum_{k=0}^{n}{}'' B_{k,n} T_k(x) \tag{2}$$

where $T_k(x) = \cos(k \arccos x)$, Chebyshev polynomial of the first kind of degree k, and the double prime on the summation sign indicates that the first and the last terms are to be halved, then the coefficients $B_{k,n}$ are given by

$$\begin{aligned} B_{k,n} &= \frac{2}{n} \sum_{i=0}^{n}{}'' f(x_i) T_k(x_i) \\ &= \frac{2}{n} \sum_{i=0}^{n}{}'' f(x_i) T_i(x_k)\,, \end{aligned} \tag{3}$$

since $T_k(x_i) = T_i(x_k)$, and $x_i = \cos(i\pi/n)$, $i = 0(1)n$. An elegant method for the evaluation of the coefficients $B_{k,n}$ is described by Clenshaw [4].

Let C be a closed contour enclosing the interval $[-1, 1]$ in its interior and let $f(z)$ be regular within C. Since the practical abscissas are the zeros of the polynomial $T_{n+1}(x) - T_{n-1}(x)$, the error $\psi_n(x)$ of the Lagrange interpolation for $f(x)$ at these abscissas can be expressed by a contour integral (Davis [5, Theorem 3.6.1]) as

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$$\psi_n(x) = \frac{[T_{n+1}(x) - T_{n-1}(x)]}{2\pi i} \int_C \frac{f(z)dz}{(z-x)[T_{n+1}(z) - T_{n-1}(z)]} \tag{4}$$

for $x \in [-1, 1]$. If n is even, the integration of (2) gives

$$\int_{-1}^{1} f(x)dx \simeq \int_{-1}^{1} \Psi_n(x)dx = \sum_{j=0}^{n/2}{}'' \frac{(-2)B_{2j,n}}{4j^2 - 1}. \tag{5}$$

Substituting for $B_{2j,n}$ from the first of the relations (3), the Clenshaw-Curtis approximate integration formula can be rewritten as

$$\int_{-1}^{1} f(x)dx \simeq \sum_{i=0}^{n}{}'' \lambda_i f(x_i) , \tag{6}$$

where the weights λ_i are given by

$$\lambda_i = \frac{2}{n} \sum_{j=0}^{n/2}{}'' \frac{(-2)T_{2j}(x_i)}{4j^2 - 1} \quad \text{for} \quad i = 0(1)n . \tag{7}$$

The error of the Clenshaw-Curtis quadrature formula is given by

$$E_n(\Psi) = \int_{-1}^{1} \psi_n(x)dx = \frac{1}{\pi i} \int_C \frac{[Q_{n+1}^*(z) - Q_{n-1}^*(z)]}{[T_{n+1}(z) - T_{n-1}(z)]} f(z)dz , \tag{8}$$

where we have put

$$Q_n^*(z) = \frac{1}{2} \int_{-1}^{1} \frac{T_n(x)dx}{z - x} . \tag{9}$$

Equation (9) defines $Q_n^*(z)$ as a single-valued analytic function in the z-plane with the interval $[-1, 1]$ deleted.

2.1. *The Quadrature Formula Based on the Classical Abscissas.* Let $\Phi_n(x)$ denote the Lagrange interpolation polynomial for the abscissas $x_i = \cos (2i + 1)\pi/(2n + 2)$, $i = 0(1)n$ which are the zeros of $T_{n+1}(x)$. The computation of the polynomial $\Phi_n(x)$ has been discussed in detail by Elliott [2].

To describe the corresponding quadrature formula, let

$$\Phi_n(x) = \sum_{k=0}^{n}{}' A_{k,n} T_k(x) , \tag{10}$$

where the prime on the summation sign indicates that the first term is to be halved. The coefficients $A_{k,n}$ are given by

$$A_{k,n} = \frac{2}{n+1} \sum_{i=0}^{n} f(x_i) T_k(x_i) \quad \text{for} \quad i = 0(1)n , \tag{11}$$

where $x_i = \cos (2i + 1)\pi/(2n + 2)$ for $i = 0(1)n$. Now, the integration of (10) gives the quadrature formula based on these abscissas as

$$\begin{aligned} \int_{-1}^{1} f(x)dx &\simeq \int_{-1}^{1} \Phi_n(x)dx \\ &\simeq \sum_{k=0}^{n}{}' A_{k,n} \left(\int_{-1}^{1} T_k(x)dx \right) . \end{aligned} \tag{12}$$

But

$$\int_{-1}^{1} T_k(x)dx = \frac{2}{1-k^2} \quad \text{if} \quad 1-k \text{ is odd}$$
$$= 0 \quad \text{if} \quad 1-k \text{ is even}.$$

Since n is even, putting $k = 2m$, we get

$$\int_{-1}^{1} f(x)dx \simeq \sum_{m=0}^{n/2}{}' \frac{(-2)A_{2m,n}}{4m^2-1}. \tag{13}$$

Substituting for $A_{2m,n}$ from (11), the above approximate integration formula can be put in the alternative form,

$$\int_{-1}^{1} f(x)dx \simeq \sum_{i=0}^{n} \mu_i f(x_i), \tag{14}$$

where the weights μ_i are given by

$$\mu_i = \frac{2}{n+1} \sum_{j=0}^{n/2}{}' \frac{(-2)T_{2j}(x_i)}{4j^2-1}, \qquad i = 0(1)n. \tag{15}$$

Let $\phi_n(x) = f(x) - \Phi_n(x)$ denote the error of interpolation for $f(x)$ at the classical abscissas. Then, $\phi_n(x)$ may be expressed in terms of a contour integral as

$$\phi_n(x) = \frac{T_{n+1}(x)}{2\pi i} \int_C \frac{f(z)dz}{(z-x)T_{n+1}(z)}. \tag{16}$$

The error $E_n(\Phi)$ for the quadrature formula (14) is given by

$$E_n(\Phi) = \int_{-1}^{1} \phi_n(x)dx = \frac{1}{\pi i} \int_C \frac{Q_{n+1}^*(z)}{T_{n+1}(z)} f(z)dz. \tag{17}$$

3. A Lemma for $Q_n^*(z)$. Introduce the ellipse $\mathcal{E}_\rho$ in the z-plane by

$$z = \tfrac{1}{2}(\xi + \xi^{-1}), \qquad \xi = \rho e^{i\theta}, \qquad 0 \leqq \theta \leqq 2\pi$$

with foci at $z = \pm 1$ and whose sum of semiaxes is ρ ($\rho > 1$).

We establish the following lemma.

LEMMA. *For $z \in \mathcal{E}_\rho$,*

$$Q_n^*(z) = \xi^{-n-1} \sum_{k=-[n/2]}^{\infty} \frac{\sigma_{n,n+2k+1}^*}{\xi^{2k}}, \tag{18}$$

where

$$\sigma_{n,n+2k+1}^* = 2(n+2k+1)/(2n+2k+1)(2k+1),$$

$[k]$ = *greatest integer* $\leqq k$.

Proof. In (9), setting $x = \cos\theta$ and transforming to the ξ-plane, we get

$$Q_n^*(z) = \xi^{-1} \int_0^{\pi} \frac{\cos(n\theta)\sin\theta d\theta}{1 - 2\cos\theta\xi^{-1} + \xi^{-2}}, \tag{19}$$

since $T_n(\cos\theta) = \cos n\theta$. Now,

$$\frac{\sin\theta}{\xi}[1 - 2\cos\theta\xi^{-1} + \xi^{-2}]^{-1} = \sum_{m=1}^{\infty} \frac{\sin m\theta}{\xi^m}. \tag{20}$$

The last series converges uniformly and absolutely for $0 \leqq \theta \leqq \pi$ and for all $|\xi| \geqq \rho > 1$. Substituting (20) in (19),

$$Q_n^*(z) = \sum_{m=1}^{\infty} \frac{\sigma_{nm}^*}{\xi^m},$$

where

$$\begin{aligned}\sigma_{nm}^* &= \int_0^\pi \cos n\theta \sin m\theta d\theta \\ &= \frac{2m}{m^2 - n^2} \quad \text{if} \quad m - n \text{ is odd} \\ &= 0 \quad \text{if} \quad m - n \text{ is even}.\end{aligned}$$

The result follows by putting $m - n = 2k + 1$ and observing that $k \geqq -[n/2]$ for $n, m = 1, 2, 3, \cdots$.

From the above lemma we deduce

COROLLARY 1. *For* $z \in \mathcal{E}_\rho$,

$$Q_{n+1}^*(z) = \xi^{-n} \sum_{k=-[(n-1)/2]}^{\infty} \frac{\sigma_{n+1,n+2k}^*}{\xi^{2k}} \tag{21}$$

where

$$\sigma_{n+1,n+2k}^* = 2(n + 2k)/(2n + 2k + 1)(2k - 1)$$

and

$$\sigma_{n+1,n+2k}^* \leqq 2n/(2n + 1).$$

COROLLARY 2. *For* $z \in \mathcal{E}_\rho$,

$$Q_{n+1}^*(z) - Q_{n-1}^*(z) = \xi^{-n} \sum_{k=-[(n-1)/2]}^{\infty} \frac{\lambda_{nk}^*}{\xi^{2k}} \tag{22}$$

where

$$\lambda_{nk}^* = 8n(n + 2k)/[4(n + k)^2 - 1][4k^2 - 1]$$

and

$$\lambda_{nk}^* \leqq 8n^2/(4n^2 - 1).$$

Proof. Subtracting (18) with n replaced by $n - 1$ from (21),

$$Q_{n+1}^*(z) - Q_{n-1}^*(z) = \xi^{-n} \sum_{k=-[(n-1)/2]}^{\infty} \frac{\lambda_{nk}^*}{\xi^{2k}}$$

where

$$\begin{aligned}\lambda_{nk}^* &= \sigma_{n+1,n+2k}^* - \sigma_{n-1,n+2k}^* \\ &= \frac{8n(n + 2k)}{[4(n + k)^2 - 1](4k^2 - 1)}.\end{aligned}$$

Also,

$$\lambda_{nk}^* \leqq |\lambda_{n,0}| = \frac{8n^2}{4n^2 - 1}.$$

4. Error Estimates. We now obtain error estimates for the Clenshaw-Curtis quadrature formula for all functions analytic on $[-1, 1]$. Simultaneously, we shall obtain estimates for $E_n(\Phi)$.

Let $f(x)$ be analytic on $[-1, 1]$. Then, for some $\rho > 1$, f can be continued analytically so as to be single valued and regular in the closure of $\mathcal{E}_\rho$. In (8), taking the contour to be an ellipse $\mathcal{E}_\rho$, we have

$$|E_n(\Psi)| \leqq \frac{1}{\pi} \int_{\mathcal{E}_\rho} \frac{|Q_{n+1}^*(z) - Q_{n-1}^*(z)|\, |f(z)|\, |dz|}{|T_{n+1}(z) - T_{n-1}(z)|}. \tag{23}$$

Now, for n even, from (22) we have

$$\begin{aligned} |Q_{n+1}^*(z) - Q_{n-1}^*(z)| &\leqq \rho^{-n}\left(\frac{8n^2}{4n^2 - 1}\right) \sum_{k=-(n/2)+1}^{\infty} \rho^{-2k} \\ &\leqq \left(\frac{8n^2}{4n^2 - 1}\right)(\rho^2 - 1)^{-1}, \end{aligned}$$

and

$$\frac{|dz|}{|T_{n+1}(z) - T_{n-1}(z)|} \leqq \rho^{-1}(\rho^n - \rho^{-n})^{-1}|d\xi|.$$

Making use of these results, from (23) we obtain the following theorem.

THEOREM 1. *Let $f(x)$ be analytic on $[-1, 1]$ and be continuable analytically so as to be single valued and regular in the closure of an ellipse $\mathcal{E}_\rho$ with foci at $z = \pm 1$ and whose sum of semiaxes is ρ $(\rho > 1)$. Then, for n even,*

$$|E_n(\Psi)| \leqq \left(\frac{16n^2}{4n^2 - 1}\right) \frac{M(\rho)}{(\rho^2 - 1)(\rho^n - \rho^{-n})} \tag{24}$$

where $M(\rho) = \max_{z \in \mathcal{E}_\rho} |f(z)|$ (or equivalently on $|\xi| = \rho$).

4.1. We next obtain an estimate for $E_n(\Phi)$. From (21), we obtain for n even,

$$\begin{aligned} |Q_{n+1}^*(z)| &\leqq \rho^{-n}\left(\frac{2n}{2n + 1}\right) \sum_{k=-(n/2)+1}^{\infty} \rho^{-2k} \\ &\leqq \left(\frac{2n}{2n + 1}\right)(\rho^2 - 1)^{-1}. \end{aligned} \tag{25}$$

Now, taking the contour to be an ellipse $\mathcal{E}_\rho$ in (17), we have

$$|E_n(\Phi)| \leqq \frac{1}{\pi} \int_{\mathcal{E}_\rho} \frac{|Q_{n+1}^*(z)|\, |f(z)|\, |dz|}{|T_{n+1}(z)|}. \tag{26}$$

Employing (25), we obtain the following theorem from (26).

THEOREM 2. *Let $f(x)$ satisfy the regularity conditions of Theorem 1. Then, for n even,*

$$|E_n(\Phi)| \leqq \left(\frac{4n}{2n + 1}\right) \frac{(\rho + \rho^{-1})}{(\rho^2 - 1)} \frac{M(\rho)}{(\rho^{n+1} - \rho^{-n-1})}. \tag{27}$$

Remarks. (i) From (24) and (27) it would appear that the estimate for $E_n(\Phi)$ is nearly half of that for $E_n(\Psi)$ for large n and $\rho \gg 1$. For small values of n and ρ near 1, the estimate (27) is still less than the estimate (24).

(ii) The estimates (24) and (27) obtained for the ellipse will be reasonably reliable for large ρ, while these estimates are poor if ρ is near 1.

(iii) For fixed n and varying ρ, a "least conservative" upper bound can be established for these estimates for some ρ^* $(1 < \rho^* \leqq \rho)$. However, observe that if $f(z)$ is entire, ρ^* will be a value of ρ for which the right side of (24) or (27) is a minimum.

5. Example. We illustrate the error estimate (24) for the Clenshaw-Curtis quadrature for the function $f(x) = 1/(x + 4)$, and compare the estimates obtained with those given by Fraser and Wilson [3].

We select $\rho = 7$ for which $f(z) = 1/(z + 4)$ is regular within the closed ellipse $\mathcal{E}_7$. Now, on $\mathcal{E}_\rho$,

$$|f(z)| \leqq M(\rho) = \frac{2\rho}{(4 + (15)^{1/2} - \rho)(\rho - (4 - (15)^{1/2}))}.$$

Thus, $M(7) \doteq 2.33333347$.

The estimate (24) for the error of the Clenshaw-Curtis quadrature applied to this function is given by

$$|E_n(\Psi)| \leqq \left(\frac{n^2}{4n^2 - 1}\right) \frac{M(7)}{(7^n - 7^{-n})}. \tag{28}$$

The exact value of $\int_{-1}^{1} dx/(4 + x) \doteq 0.5108\ 2562$.

TABLE I

n	*estimated error*	*actual error*	*estimates of Fraser-Wilson*
2	0.0042 3456	0.0002 8549	0.0166 6667
4	0.0000 8230	0.0000 0125	0.0002 6882
8	0.0000 00027	0.0000 0000	0.0000 0007

The error estimated by (28) is compared in Table I with the actual error and the estimates given by Fraser and Wilson [3].

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