Error Bounds for Gauss-Chebyshev Quadrature*

By Franz Stetter

1. Introduction. For the error $E(f)$ of the numerical quadrature

$$E(f) = \int_a^b f(x)dx - \sum_{k=1}^{N} a_k f(x_k) ,$$

Davis [1] was the first to give bounds of the kind $\sigma\|f\|$ which do not involve derivatives of the function f, but f is assumed to be analytic in a region containing the interval $[a, b]$. Since then such estimates have been developed in various directions, e.g., different norms of f, influence of the interval length, or optimal choice of the coefficients a_k and x_k.

In this paper, we show that similar bounds can also be derived for quadrature rules based on suitable weight functions $w(x)$. We especially consider Gaussian rules with the weight $w(x) = (1 - x^2)^{-1/2}$ over the interval $[-1, 1]$:

$$R(f) = \int_{-1}^{1} \frac{f(x)}{(1-x^2)^{1/2}}\,dx - \frac{\pi}{N}\sum_{\nu=1}^{N} f\left(\cos\left(\frac{2\nu-1}{2N}\pi\right)\right). \tag{1.1}$$

In this connection we also refer to Stenger [4] who gives general error developments.

2. Bounds. Let f be analytic in $|z| \leqq r$, $r > 1$. Applying the linear and continuous operator R to the Cauchy integral of f

$$f(x) = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z-x}\,dz = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(re^{i\phi})}{1 - xr^{-1}e^{-i\phi}}\,d\phi \tag{2.1}$$

we therefore immediately get by means of the Cauchy-Schwarz inequality:

$$|R(f)|^2 \leqq \left\{\frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{|R(x^n)|^2}{r^{2n}}\right\}\left\{\int_0^{2\pi}|f(re^{i\phi})|^2 d\phi\right\} = \sigma^2\|f\|^2 , \tag{2.2}$$

where σ^2 depends on N and r. (r is to be chosen so that $\sigma^2\|f\|^2$ is minimal.) Now $R(x^n) = 0$ for $n = 0, 1, \cdots, 2N - 1$ (the rule (1.1) is exact for polynomials of degree less than $2N$) and $R(x^{2n+1}) = 0$ for $n = 0, 1, \cdots$ ((1.1) is symmetric). From

$$\int_{-1}^{1}\frac{x^{2n}}{(1-x^2)^{1/2}}\,dx = \pi\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} = \pi\frac{(2n-1)!!}{(2n)!!}, \tag{2.3}$$

we obtain the expression

$$\sigma^2 = \frac{\pi}{2}\sum_{n=N}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!} - \frac{1}{N}\sum_{\nu=1}^{N}\cos^{2n}\left(\frac{2\nu-1}{2N}\pi\right)\right)^2 r^{-4n} . \tag{2.4}$$

(a) *Case* $N = 1$. Since $\cos \pi/2 = 0$ we get

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$$\sigma^2 = \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} \right)^2 r^{-4n} < \frac{\pi}{8} \frac{1}{r^4 - 1}, \tag{2.5}$$

and hence,

$$|R(f)| \leqq \frac{1}{2} \left(\frac{\pi}{2(r^4 - 1)} \right)^{1/2} \|f\| . \tag{2.6}$$

(b) *Case* $N = 2$. Now

$$\sigma^2 = \frac{\pi}{2} \sum_{n=2}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} - \frac{1}{2^n} \right)^2 r^{-4n} . \tag{2.7}$$

From

$$\operatorname*{Max}_n \left(\frac{(2n-1)!!}{(2n)!!} - \frac{1}{2^n} \right)^2 = \left(\frac{55}{256} \right)^2$$

it follows that

$$|R(f)| \leqq \frac{55}{256 r^2} \left(\frac{\pi}{2(r^4 - 1)} \right)^{1/2} \|f\| . \tag{2.8}$$

(c) *Case* $N = 3$. $|R(x^{2n})|$ assumes its maximum for $n = 12$; the value is

$$0.14006 \cdots < 1/7 .$$

Hence

$$|R(f)| < \frac{1}{7 r^4} \left(\frac{\pi}{2(r^4 - 1)} \right)^{1/2} \|f\| . \tag{2.9}$$

(d) *Case* $N \geqq 4$.

In the theory of numerical integration, it is shown that the error $R(f)$ can be expressed by $R(f) = af^{(2N)}(\xi)$ where $a > 0$ and $-1 \leqq \xi \leqq 1$; hence (for $n \geqq N$),

$$\begin{aligned} 0 \leqq R(x^{2n}) &= \pi \left(\frac{(2n-1)!!}{(2n)!!} - \frac{1}{N} \sum_{\nu=1}^{N} \cos^{2n}\left(\frac{2\nu - 1}{2N} \pi \right) \right) \\ &\leqq \pi \frac{(2n-1)!!}{(2n)!!} \leqq \pi \frac{(2N-1)!!}{(2N)!!} . \end{aligned}$$

Thus, we get the general estimate

$$|R(f)| \leqq \frac{(2N-1)!!}{(2N)!! r^{2N-2}} \left(\frac{\pi}{2(r^4 - 1)} \right)^{1/2} \|f\| . \tag{2.10}$$

The bound (2.10) is also valid for $N = 1, 2, 3$. Using Stirling's formula for $n!$, we get from (2.10)

$$|R(f)| \leqq \frac{1.05}{r^{2N-2}(2N)^{1/2}} \left(\frac{1}{r^4 - 1} \right)^{1/2} \|f\| \tag{2.11}$$

for every $N \geqq 1$.

3. Example. Let $f(x) = x^6$. Then the two-point rule ($N = 2$) has the bound

$$|R(f)| \leqq \frac{55}{256} \frac{\pi r^4}{(r^4 - 1)^{1/2}}$$

which is minimum when $r^4 = 2$. Thus, $|R(f)| \leqq 55\pi/128$. The exact error is $3\pi/16$.

4. Remarks. Similar results can be derived by using, instead of the norm used in this paper, polynomials orthogonal over the region $|z| < r$ or orthogonal over (or on) the ellipse whose foci are ± 1. For details we refer to Davis [2].

As mentioned by Hämmerlin [3] and Stenger [4], the norm $\|f\|$ can be replaced by $\|f - P_{2N-1}\|$ where P_{2N-1} is a polynomial of degree $2N - 1$.

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1. P. Davis, "Errors of numerical approximation for analytic functions," *J. Rational Mech. Anal.*, v. 2, 1953, pp. 303–313. MR **14**, 907.
2. P. Davis, *Interpolation and Approximation*, Blaisdell, New York, 1963. MR **28** #393.
3. G. Hämmerlin, "Über ableitungsfreie Schranken für Quadraturfehler. II: Ergänzungen und Möglichkeiten zur Verbesserung," *Numer. Math.*, v. 7, 1965, pp. 232–237. MR **32** #1899.
4. F. Stenger, "Bounds on the error of Gauss-type quadratures," *Numer. Math.*, v. 8, 1966, pp. 150–160. MR **33** #5120.