Powers of a Matrix of Special Type

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In this paper it is shown that if $E^TE = I$, i.e., if the columns of E are orthonormal, and K is a diagonal matrix with all terms positive, then

$$[I + E(K - I)E^T]^n = I + E(K^n - I)E^T \tag{1}$$

for any real n. Since given any matrix V and a positive definite matrix G it is possible to find an E and K as just described satisfying

$$VGV^T = E(K - I)E^T: \tag{2}$$

this provides a method for finding any power or root of matrices of the type

$$B = I + V\,G\,V^T\,. \tag{3}$$

This becomes particularly useful for work on high speed digital computers when G is very small. For suppose V—and hence also E—is $n \times r$ with $n \gg r$ and G is $r \times r$. Then keeping only E and, trivially, the diagonal of K, in fast-access storage and using only r core locations as working storage one can perform a rapid multiplication of an $n \times 1$ vector by any of the matrices B, $B^{1/2}$, $B^{-1/2}$, B^{-1}, etc., with B as in (3).

Equation (1) can be easily proved from the identity*

$$[I + E(K^p - I)E^T][I + E(K^q - I)E^T] = I + E(K^{p+q} - I)E^T \tag{4}$$

which can be established by merely multiplying out the terms on the left side and making use of the relation $E^TE = I$.

The conversion indicated in Eq. (2) can be accomplished by orthogonalizing the columns of V by elementary column operations [1]—the round-off error problem [2], not being significant when r is small—to obtain the matrix O such that

$$O^TO = I\,. \tag{5}$$

Let the product of the corresponding elementary matrices be the matrix R, i.e.,

$$O = VR \qquad \text{or} \tag{6}$$

$$V = OR^{-1}\,. \tag{7}$$

Then let

$$H = R^{-1}GR^{-1T}\,. \tag{8}$$

Note that H is $r \times r$ and symmetric positive definite. One then solves the small eigenproblem

$$HX = XL \tag{9}$$

for X and the diagonal matrix L. It follows that

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* The author is indebted to the referee for this equation.

$$X^T X = I \,, \tag{10}$$

$$X^T = X^{-1} \,. \tag{11}$$

Then, using Eqs. (7) through (11)

$$VGV^T = OHO^T = OXLX^TO^T = ELE^T \tag{12}$$

where

$$E = OX \,. \tag{13}$$

Furthermore $E^TE = X^TO^TOX = X^TX = I$ using Eqs. (13), (5), and (10).

An alternative method, though somewhat longer but more stable in terms of rounding errors, is to perform a Cholesky factorization [3] of the positive definite matrix G, obtaining

$$G = UU^T \,. \tag{14}$$

Then form the matrix

$$C = U^TV^TVU \tag{15}$$

and solve the small $r \times r$ eigenproblem

$$CY = YL \tag{16}$$

for Y and L. Since C is positive definite all terms of L are positive and one may set

$$E = VUYL^{-1/2} \tag{17}$$

so that

$$E^TE = L^{-1/2}Y^TU^TV^TVUYL^{-1/2} = L^{-1/2}Y^TCYL^{-1/2} = L^{-1/2}LL^{-1/2} = I \tag{18}$$

and

$$ELE^T = VUYL^{-1/2}LL^{-1/2}Y^TU^TV^T = VGV^T \tag{19}$$

as desired. Then let

$$K = I + L \tag{20}$$

for K as in Eq. (2).

Another identity related to (4), though not so general, for M and K diagonal matrices of nonzero terms and F such that $F^TMF = I$, is

$$[M + MF(K - I)F^TM][M^{-1} + F(K^{-1} - I)F^T] = I \tag{21}$$

which again can be proved by simply multiplying out the terms of the product.

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2. J. H. Wilkinson, *Rounding Errors in Algebraic Processes*, Prentice-Hall, Englewood Cliffs, N. J., 1963. MR **28** #4661.
3. A. Ralston & H. S. Wilf, *Mathematical Methods for Computers*, Vol. 2, Wiley, New York, 1967, p. 71.